Cardinal Invariants and the Borel Tukey order

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Cardinal invariants of the continuum

These are quantities in the interval $[\aleph_1, 2^{\aleph_0}]$.

Examples

- $\mathfrak{b}$ = the least size of a $\leq^*$-unbounded family in ω^ω
- $\mathfrak{d}$ = the least size of a $\leq^*$-dominating family in ω^ω
- $\mathfrak{s}$ = the least size of a splitting family in $[\omega]^\omega$

(here, we say that A splits B if $|A \cap B| = |A^c \cap B| = \infty$)

Question

Each model of ZFC will exhibit a pattern of values of these cardinals. What patterns are possible?

ZFC relationships

Example

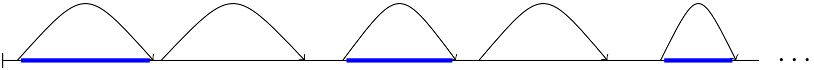
It is clear that $\mathfrak{d} \geq \mathfrak{b}$, since dominating families are necessarily unbounded.

Example

It is less clear that $\mathfrak{d} \geq \mathfrak{s}$...

Proof.

Given a dominating family $\mathcal{F}$ and an element $f \in \mathcal{F}$, find disjoint arcs:



Let B_f consist of every other region. It is not hard to see that $\{ B_f : f \in \mathcal{F} \}$ form a splitting family. □

A categorical approach

The key idea, due to Vojtáš, is to focus on the defining relations such as dominating and splitting.

Definition

Let $R \subset A \times B$ be a binary relation...

- We say that a family $\mathcal{F} \subset B$ is R-dominating if

$$\forall a \in A \;\; \exists b \in \mathcal{F} \;\; a \; R \; b \, .$$

- The dominating number of R: the least size of an R-dominating family.

Examples

- $\mathfrak{d}$ is the dominating number of the relation $\leq^*$
- $\mathfrak{s}$ is the dominating number of the relation "is split by"

Tukey morphisms

Definition

Let $R \subset A \times B$ and $R' \subset A' \times B'$ be relations. A Tukey morphism from R to R' is a map:

$$\begin{array}{ccc} B & \xrightarrow{\psi} & B' \\ R \uparrow\!\wr & & \uparrow\!\wr\, R' \\ A & \xleftarrow[\phi]{} & A' \end{array}$$

such that

$$\phi(a) \; R \; b \implies a \; R' \; \phi(b) \; .$$

In particular, ψ sends R-dominating families to R'-dominating families.

Definability of Tukey morphisms

It was hoped that Tukey morphisms would capture the kind of close combinatorial connection seen in the proof of $\mathfrak{d} \geq \mathfrak{s}$. Unfortunately:

Fact

If CH holds, then there exist Tukey morphisms between all reasonable cardinal invariants.

This led Blass to impose some definability conditions on the relations and morphisms involved:

Definition

A Tukey morphism (ϕ, ψ) from R to R' is said to be Borel if R and R' are defined on standard spaces and ϕ and ψ are Borel functions.

Question

Describe the ordering on relations $\geq_{BT}$ induced by the Borel Tukey morphisms.

Special treatment for cardinals like $\mathfrak{p}$

Example

- $\mathfrak{p}$ = the least size of a centered family with no pseudo-intersection

$\not\subset^*$-dominating

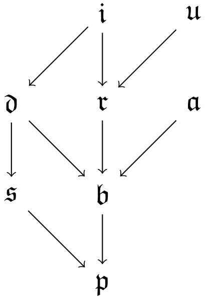
an extra condition

Definition

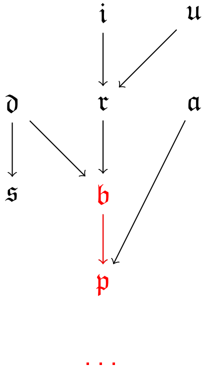
- If R is a relation and P is a property then we let $\|R\|_P$ be the least size of an R-dominating family with property P.
- A Tukey morphism from R, P to R', P' must additionally send families with property P to families with property P'.

Comparing commonly considered combinatorial cardinals

ZFC-provable inequalities ($\geq$)



The Borel Tukey order ($\geq_{BT}$)



About $\mathfrak{b}$ versus $\mathfrak{p}$

The usual proof that $\mathfrak{b} \geq \mathfrak{p}$:

- Given an unbounded family $\mathcal{F} = \mathcal{F} = \{ f_\alpha : \alpha < \kappa \}$
- Recursively arrange for $\mathrm{im}(f_\alpha) \supset^* \mathrm{im}(f_\beta)$
- Then $\{ \mathrm{im}(f_\alpha) \}$ has no pseudo-intersection; if A were such the enumeration g_A of A would be a bound for $\{ f_\alpha \}$.

Of course, this argument does not give a morphism.

Theorem

- *There is no continuous Tukey morphism from* $\mathfrak{b}$ *to* $\mathfrak{p}$
- *There is a Borel Tukey morphism from* $\mathfrak{b}$ *to* $\mathfrak{p}$

Chains

Definition

A family $\mathcal{F} \subset [\omega]^\omega$ is said to be n-splitting if for all $A_1, \ldots, A_n$ there exists $B \in \mathcal{F}$ which splits them all.

Remark

The n-splitting relations S_n all define the same cardinal, $\mathfrak{s}$.

Theorem

We have $S_1 <_{BT} S_2 <_{BT} S_3 <_{BT} \cdots$

(This mirrors a result of Mildenberger who did the same for a family of unsplitting relations.)

Antichains

Definition

A family $\mathcal{F} \subset [\omega]^\omega$ is said to be n, m-splitting if for all $A_1, \dots, A_n$ there exists $B \in \mathcal{F}$ which splits m of them.

Let $S_{n,m}$ denote the n, m-splitting relation.

Theorem

If $n_i/m_i \to \infty$ sufficiently rapidly, then for $i \neq j$ we have $S_{n_i,m_i} \perp_{BT} S_{n_j,m_j}$.