On dimension and Borel reducibility

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Standard Borel spaces

Many classification problems can be presented as equivalence relations on standard Borel spaces.

Definition

A standard Borel space is a Polish space equipped just with its σ-algebra of Borel sets.

Examples

- Any Borel subset of a Polish space X
- The space $F(X)$ of closed subsets of a Polish space X

Countable groups

Many classification problems can be presented as equivalence relations on standard Borel spaces.

Example

The classification problem for countable groups.

Definition

- Let $X_{\mathcal{G}}$ denote the space of countable groups (think of it as a Borel subset of $\mathcal{P}(\omega^3)$)
- Let $\cong_{\mathcal{G}}$ denote the isomorphism equivalence relation on $X_{\mathcal{G}}$

Complexity of group isomorphism

- Let $X_{\mathcal{G}}$ denote the space of countable groups (think of it as a Borel subset of $\mathcal{P}(\omega^3)$)

Observation

Groups $G, G' \in X_{\mathcal{G}}$ are isomorphic iff there exists $f : \omega \to \omega$ carrying the group operation of G to that of G'.

Remark

This is a $\mathbf{\Sigma}^1_1$ definition.

Theorem (Mekler)

In fact, $\cong_{\mathcal{G}}$ is a $\mathbf{\Sigma}^1_1$-complete set of pairs, and hence it is not Borel.

Torsion-free abelian groups of finite rank

Definition

If A is a torsion-free abelian group, then the rank of A is the size of a maximal $\mathbb{Z}$-independent set.

Fact

Any torsion-free abelian group of rank n is isomorphic to a subgroup of $\mathbb{Q}^n$.

Definition

- Let TFA_n denote the space of rank n subgroups of $\mathbb{Q}^n$
- Let $\cong_n$ denote the isomorphism equivalence relation on TFA_n

Complexity of $\cong_n$

- Let TFA_n denote the space of rank n subgroups of $\mathbb{Q}^n$

Observation

Subgroups $A, B \leq \mathbb{Q}^n$ are isomorphic iff there exists $g \in \mathrm{GL}_n(\mathbb{Q})$ such that $B = g(A)$.

Remark

This is a countable quantifier; it follows that $\cong_n$ is Borel.

The Borel/non-Borel distinction is useful, but we have a finer notion of complexity in mind...

Smooth equivalence relations

Definition

The equivalence relation E on X is called smooth (or completely classifiable) iff there exists a standard Borel space $\mathcal{I}$ of invariants and a Borel function $f\colon X \to \mathcal{I}$ such that

$$x \, E \, x' \iff f(x) = f(x') \, .$$

The map f tells you how to find complete invariants for the classification problem up to E.

Example

The isomorphism problem for countable divisible groups is smooth. Just let $f(A) = \langle n_0, n_2, n_3, n_5, \ldots \rangle$, where

$$A \cong \mathbb{Q}^{n_0} \oplus \mathbb{Z}(2^\infty)^{n_2} \oplus \mathbb{Z}(3^\infty)^{n_3} \oplus \mathbb{Z}(5^\infty)^{n_5} \oplus \cdots$$

Borel reducibility

Definition (H. Friedman–Stanley)

Let E, F be equivalence relations on standard Borel spaces X, Y. We say that E is Borel reducible to F (written $E \leq_B F$) iff there exists a Borel function $f\colon X \to Y$ satisfying

$$x \, E \, x' \iff f(x) \, F \, f(x') \, .$$

We say that f is a Borel reduction from E to F.

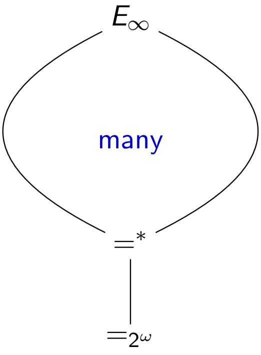
Example

$\cong_n \leq_B \cong_{n+1}$ via the map $A \mapsto A \oplus \mathbb{Q}$.

Countable Borel equivalence relations

Definition

The Borel equivalence relation E is said to be countable iff every E-class is countable.



e.g., locally finite graphs, f.g. groups

e.g., torsion-free abelian groups of finite rank

almost equality on 2^ω

equality on 2^ω

Big questions of the 90s

Question

Are there infinitely many countable Borel equivalence relations up to bireducibility?

Definition

E, F are Borel bireducible (written $E \sim_B F$) iff $E \leq_B F$ and $F \leq_B E$.

Question

Are there infinite chains? Antichains?

Problem

Describe the structure of the (pre) partial order $\leq_B$ on the countable Borel equivalence relations.

Orbit equivalence relations

Definition

Let X be a standard Borel space and $\Gamma \curvearrowright X$ the Borel action of some countable group. The orbit equivalence relation E_Γ is defined by

$$x \, E_\Gamma \, x' \iff \Gamma x = \Gamma x' \, .$$

Example

The isomorphism relation $\cong_n$ on the subspace $\mathrm{TFA}_n \subset \mathcal{P}(\mathbb{Q}^n)$ of torsion-free abelian groups of rank n is induced by the action $\mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathrm{TFA}_n$.

Theorem (Feldman–Moore)

If E is any countable Borel equivalence relation on X, then there exists a countable group Γ and a Borel action $\Gamma \curvearrowright X$ such that $E = E_\Gamma$.

Rigidity for countable Borel equivalence relations

Theorem (Adams–Kechris)

There exists an uncountable family of pairwise Borel incomparable countable Borel equivalence relations.

Idea

Use the concept of rigidity: in special cases, groups which are highly incompatible give rise to orbit spaces which are highly incompatible.

More precisely, Adams–Kechris used the following consequence of Zimmer's cocycle superrigidity theorem:

Theorem (Adams–Kechris)

Let $\Gamma_i \curvearrowright X_i$ be free, ergodic actions of lattices in higher rank, connected, centerless, simple Lie groups G_i. If $E_{\Gamma_0} \leq_B E_{\Gamma_1}$, then there exist $N \trianglelefteq H \leq G_1$ such that $G_0 \cong H/N$.

TFA history

1937 Baer showed that $\cong_1$ lies at the level of $=^*$.

1938 Kurosh and Malcev "classified" the rank 2 and higher groups by invariants consisting of a sequence of p-adic matrices modulo certain operations.

1998 Hjorth proved that in fact $\cong_2$ is strictly more complex than $\cong_1$.

Question

Do the $\cong_n$ increase strictly in complexity beyond $n = 2$?

Try to use rigidity for the TFA problem

Suppose that f is a Borel reduction from $\cong_{n+1}$ to $\cong_n$.

Idea

Recall that $\cong_n$ is induced by the action $\mathrm{GL}_n(\mathbb{Q}) \curvearrowright \mathrm{TFA}_n$. Try to use Adams–Kechris to reach a contradiction.

Theorem (Adams–Kechris)

Let $\Gamma_i \curvearrowright X_i$ be free ergodic actions of lattices in higher rank, connected centerless simple Lie groups G_i. If $E_{\Gamma_0} \leq_B E_{\Gamma_1}$, then there exist $N \trianglelefteq H \leq G_1$ such that $G_0 \cong H/N$.

Theorem (Hjorth)

There exists an ergodic, $\mathrm{SL}_n(\mathbb{Z})$-invariant measure on TFA_n.

A chain of TFAs

Theorem (Adams–Kechris)

Let $\cong_n^*$ denote the restriction of $\cong_n$ to the subspace $S(n) \subset \mathrm{TFA}_n$ of rigid TFA*s of rank n. Then:*

$$\cong_2^* \ <_B \ \cong_3^* \ <_B \ \cong_4^* \ <_B \ \cdots$$

Definition

A subgroup $A \leq \mathbb{Q}^n$ is said to be rigid iff $\mathrm{Aut}(A) = \{ \pm Id \}$.

More chains of TFAs

Thomas completed the Hjorth/Adams/Kechris analysis to obtain:

Theorem (Thomas)

$$\cong_2 \ <_B \ \cong_3 \ <_B \ \cong_4 \ <_B \ \cdots$$

Other cases where the complexity increases strictly with the rank:

- Dimension groups
- p-local groups (next slide)
- TFAs, but considered up to quasi-isomorphism

Definition

Subgroups $A, B \leq \mathbb{Q}^n$ are said to be quasi-isomorphic iff B is commensurable with an isomorphic copy of A.

Antichains of local TFAs

Definition

An abelian group is said to be p-local iff it is q-divisible for every $q \neq p$.

Definition

Let $\cong_n^{(p)}$ denote the restriction of $\cong_n$ to the subspace $\mathrm{TFA}_n^{(p)}$ of p-local torsion-free abelian groups of rank n.

Theorem (Thomas)

If $n \geq 2$ and $p \neq q$, then $\cong_n^{(p)}$ is Borel incomparable with $\cong_n^{(q)}$.

The main theorem statement

Question (Thomas)

What role does "dimension" play in deciding whether $E \leq_B F$?

Lemma ( **)**

Suppose that $3 \leq m < n$ *and that* $p \neq q$*. Then* $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{SL}_m(\mathbb{Z}_p)$ *is Borel incomparable with* $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{Z}_q)$*.*

Theorem ( **)**

Suppose that $3 \leq m < n$ *and that* $p \neq q$*. Then* $\cong_m^{(p)}$ *is Borel incomparable with* $\cong_n^{(q)}$*.*

$\longrightarrow$ So the locality prime can be used as an invariant, regardless of the dimension.

Proving the lemma

Lemma ()

Suppose that $3 \leq m < n$ and that $p \neq q$. Then $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{SL}_m(\mathbb{Z}_p)$ is Borel incomparable with $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{Z}_q)$.

Proof sketch

Suppose, towards a contradiction, that $f\colon \mathrm{SL}_m(\mathbb{Z}_p) \to \mathrm{SL}_n(\mathbb{Z}_p)$ is a Borel reduction from $E_{\mathrm{SL}_m \mathbb{Z}}$ to $E_{\mathrm{SL}_n \mathbb{Z}}$.

Idea

Use rigidity to replace f by a map which not only takes orbits to orbits, but also carries one action to the other.

What we seek from ergodic theory

Definition

Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$.

- Borel homomorphism from E_Γ to E_Λ: A Borel function $f \colon X \to Y$ such that
$$\Gamma x = \Gamma x' \implies \Lambda f(x) = \Lambda f(x')$$
- Permutation group homomorphism from $\Gamma \curvearrowright X$ to $\Lambda \curvearrowright Y$: a Borel homomorphism f together with a group homomorphism $\phi \colon \Gamma \to \Lambda$ such that
$$f(\gamma x) = \phi(\gamma) f(x)$$

We want to replace a Borel homomorphism with a permutation group homomorphism.

Superrigidity for profinite actions

Theorem (A. Ioana)

And suppose that f is a Borel homomorphism from E_Γ to E_Λ, and that the following hypotheses are satisfied.

- Γ *has property* (T)
- $\Gamma \curvearrowright X$ *is profinite, free, and ergodic*
- $\Lambda \curvearrowright Y$ *is free*

Then (after a finite error), f is equivalent to a permutation group homomorphism from $\Gamma \curvearrowright X$ to $\Lambda \curvearrowright Y$.

Remark

We are interested in the action $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{SL}_m(\mathbb{Z}_p)$; this satisfies all of the hypotheses on $\Gamma \curvearrowright X$.

The conclusion of the proof

Dense subgroups of compact groups

We have: a permutation group homomorphism (ϕ, f) from $\mathrm{SL}_m(\mathbb{Z}) \curvearrowright \mathrm{SL}_m(\mathbb{Z}_p)$ to $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{Z}_q)$.

Lemma (Gefter, Furman)

Suppose that K_0, K_1 are compact groups and $\Gamma_i \leq K_i$ are dense subgroups. Let (ϕ, f) be a permutation group homomorphism from $\Gamma_0 \curvearrowright K_0$ to $\Gamma_1 \curvearrowright K_1$. Then (off of a null set) f an affine mapping.

Definition

$f : K_0 \to K_1$ is said to be an affine mapping iff $f(k) = \Phi(k)t$ for some homomorphism $\Phi \colon K_0 \to K_1$ and $t \in K_1$.

In particular there exists a homomorphism $\mathrm{SL}_m(\mathbb{Z}_p) \to \mathrm{SL}_n(\mathbb{Z}_q)$; this is a contradiction! $\square$

Deriving the main theorem

Theorem ( **)**

Suppose that $3 \leq m < n$ and that $p \neq q$. Then $\cong_m^{(p)}$ is not Borel reducible to $\cong_n^{(q)}$.

The main ingredient is the Kurosh-Malcev classification:

Theorem (Kurosh-Malcev)

The map $A \mapsto A \otimes \mathbb{Z}_p$ is a $\mathrm{GL}_n(\mathbb{Q})$-preserving isomorphism between $\mathsf{TFA}_n^{(p)}$ and the space of $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^n$.

Lemma

$\mathrm{SL}_n(\mathbb{Z}_p)$ acts on the space of $\mathbb{Z}_p$-submodules of $\mathbb{Q}_p^n$, and any of its orbits meets every $\mathrm{GL}_n(\mathbb{Q})$-orbit.

One applies the methods we have outlined to the action of $\mathrm{SL}_n(\mathbb{Z})$ on some $\mathrm{SL}_n(\mathbb{Z}_p)$ orbit, which is a transitive $\mathrm{SL}_n(\mathbb{Z}_p)$-space.