The classification of torsion-free abelian groups up to isomorphism and quasi-isomorphism

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Torsion-free abelian groups of finite rank

A concise and selective history

1937 Baer classified the torsion-free abelian groups of rank 1.

1998 Hjorth proved that the classification problem for torsion-free abelian groups of rank 2 is strictly more complex than that for the rank 1 groups.

2001 Thomas proved that the problem for rank $n+1$ groups is strictly more complex than the problem for rank n.

Question

What does it mean for one classification problem to be strictly more complex than another?

Standard Borel spaces

Definition

A standard Borel space is a Polish space X equipped just with its σ-algebra of Borel sets.

Example

$\mathbb{R}, \mathbb{Q}_p, \mathcal{P}(\mathbb{N})$, Borel subsets of these

Example

The space TFA_n of torsion-free abelian groups of rank n.

This is the standard Borel space consisting of those $A \in \mathcal{P}(\mathbb{Q}^n)$ which are subgroups of $\mathbb{Q}^n$ of rank n.

Remark

Now, studying the classification problem for torsion-free abelian groups of rank n amounts to studying the isomorphism equivalence relation on TFA_n.

Borel reducibility of equivalence relations

Definition

Let E, F be equivalence relations on standard Borel spaces X, Y. Then E is Borel reducible to F (written $E \leq_B F$) iff there exists a Borel map $f : X \to Y$ satisfying:

$$a \; E \; b \iff f(a) \; F \; f(b)$$

Meaning. . .

- Any set of invariants for F can be used as invariants for E.
- The E-classification problem on X is no harder than the F-classification problem on Y.

Example of a Borel reduction

Torsion-free abelian groups

Definition

Let $\cong_n$ be the isomorphism equivalence relation on the space TFA_n of torsion-free abelian groups of rank n.

Fact

$\cong_n \leq_B \cong_{n+1}$

Proof.

Use the map $A \mapsto A \oplus \mathbb{Q}$. □

Hjorth's 1998 theorem and Thomas's 2001 theorem

Theorem

The classification problem for torsion-free abelian groups of rank n increases strictly in complexity with the rank n. In symbols:

$$\cong_1 \;<_B\; \cong_2 \;<_B\; \cong_3 \;<_B\; \cdots \;<_B\; \cong_n \;<_B\; \cdots$$

(The first $<_B$ is Hjorth's part.)

Quasi-isomorphism

Definition

Subgroups $A, B \leq \mathbb{Q}^n$ are said to be quasi-isomorphic (written $A \sim_n B$) iff A and B have isomorphic subgroups of finite index.

Thomas found the quasi-isomorphism relation simpler to work with and initially proved:

Theorem (Thomas, 2001)

$$\sim_1 \ <_B \ \sim_2 \ <_B \ \sim_3 \ <_B \ \cdots \ <_B \ \sim_n \ <_B \ \cdots$$

Isomorphism versus quasi-isomorphism

The question

Theorem (Corner)

There exists a torsion-free abelian group A of rank 3 such that

$$A_1 \oplus A_2 \;\cong\; A \;\cong\; B_1 \oplus B_2 \oplus B_3$$

and A_i, B_j are indecomposable!

Theorem (Jónsson)

There is unique decomposition of torsion-free abelian groups in the quasi-isomorphism category.

Question

Is quasi-isomorphism simpler ($\leq_B$) than isomorphism?

Isomorphism versus quasi-isomorphism

The answer

Answer

Isomorphism and quasi-isomorphism of p-local torsion-free abelian groups of rank n are incomparable, meaning that there is not a Borel reduction either way.

Definition

Let p be a prime. Then $A \leq \mathbb{Q}^n$ is p-local iff it is infinitely q-divisible for every $q \neq p$.

Conjecture

The same is true for isomorphism and quasi-isomorphism on the space of all torsion-free abelian groups of rank n.

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Simon Thomas, Rutgers University

A descriptive view of geometric group theory

Wednesday 1pm (room 1A)