the tvc and metric structures

second vaught's conjecture workshop
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Overview

- Vaught's conjecture has numerous model-theoretic, dynamical, and descriptive analogs.
- The $\mathcal{L}_{\omega_1\omega}$-Vaught conjecture and the topological Vaught conjecture are connected by López-Escobar's theorem, a cornerstone result linking countable model theory, descriptive set theory, and logic.
- This talk will feature a generalization of Lopez-Escobar's theorem for metric structures.
- As a consequence we will obtain a new model-theoretic characterization of the topological Vaught conjecture.

The space of countable structures

Definition

If $\mathcal{L}$ is a countable relational language with symbols R_i of arity n_i, then we define the space of countable $\mathcal{L}$-structures

$$\mathrm{Mod}(\mathcal{L}) = \prod \mathcal{P}(\mathbb{N}^{n_i}).$$

Recall

For $\mathcal{L}$ a language, $\mathcal{L}_{\omega_1\omega}$ denotes the extension of first-order logic in which countable conjunctions and disjunctions are allowed. (Note: We require formulas to have finitely many free variables.)

$\mathcal{L}_{\omega_1\omega}$-VC, Vaught's conjecture for $\mathcal{L}_{\omega_1\omega}$

For any sentence ϕ of $\mathcal{L}_{\omega_1\omega}$, the subset $\mathrm{Mod}(\phi) \subset \mathrm{Mod}(\mathcal{L})$ consisting of the models of ϕ meets either countably many or perfectly many isomorphism classes.

The logic action

Recall

The Polish group S_∞ of permutations of $\mathbb{N}$ acts naturally on $\mathrm{Mod}(\mathcal{L})$ by shifting the subsets of $\mathbb{N}^{n_i}$; this is called the logic action.

Remark

The logic action makes $\mathrm{Mod}(\mathcal{L})$ into an S_∞-space, and its orbits are precisely the isomorphism classes.

This action plays a key role in descriptive set theory.

Proposition (Becker–Kechris)

If $\mathcal{L}$ contains symbols of arbitrarily large arity, then $\mathrm{Mod}(\mathcal{L})$ is a universal Borel S_∞-space. That is for any Borel action $S_\infty \curvearrowright X$ there is a Borel S_∞-embedding $i\colon X \hookrightarrow \mathrm{Mod}(\mathcal{L})$.

Dynamical Vaught variants

The role of the logic action leads one to the dynamical Vaught conjectures:

TVC(S_∞), the topological Vaught conjecture for S_∞

Any standard Borel S_∞-space has countably many or perfectly many orbits.

TVC(G), for G a Polish group

Any standard Borel G-space has countably many or perfectly many orbits.

TVC

TVC(G) holds for any polish G

A connection between VC and TVC

The topological Vaught conjecture is stronger than the Vaught conjecture.

Proposition

TVC(S_∞) *implies the* $\mathcal{L}_{\omega_1\omega}$*-VC.*

Proof.

This is simply because Mod(ϕ) is an *example* of a standard Borel S_∞-space. □

Question

Can we go beyond $\mathcal{L}_{\omega_1\omega}$?

The key result: López-Escobar's theorem

Question

Can we go beyond $\mathcal{L}_{\omega_1\omega}$?

López-Escobar's theorem states that this is not necessary.

Theorem (Edgar George Kenneth López-Escobar)

If $X \subset \mathrm{Mod}(\mathcal{L})$ *is Borel and isomorphism-closed then there exists a sentence* ϕ *of* $\mathcal{L}_{\omega_1\omega}$ *such that* $X = \mathrm{Mod}(\phi)$*.*

This allows us to show that the Proposition was tight.

Corollary

$\mathrm{TVC}(S_\infty)$ *is equivalent to the* $\mathcal{L}_{\omega_1\omega}$*-VC.*

Proof of the corollary

Corollary

TVC(S_∞) *is equivalent to the* $\mathcal{L}_{\omega_1\omega}$*-VC.*

Proof.

($\Rightarrow$) This was because Mod(ϕ) is Borel.

($\Leftarrow$) Let X be a standard Borel S_∞-space.

- There exists $\mathcal{L}$ and a Borel S_∞-embedding $i\colon X \hookrightarrow \mathrm{Mod}(\mathcal{L})$. Note that $i(X)$ is Borel and isomorphism-closed.
- By López-Escobar's theorem there exists a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ such that $i(X) = \mathrm{Mod}(\phi)$.
- By the $\mathcal{L}_{\omega_1\omega}$-VC, the image $i(X)$ has countably or perfectly many isomorphism types.
- Hence X has countably many or perfectly many orbits. □

Idea of Vaught's proof of López-Escobar's theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ lies in the Borel hierarchy then X is approximated by simpler sets. Unfortunately the simpler sets will not be isomorphism-closed. We thus look for a stronger statement which applies even to sets X which are not isomorphism-closed.

Definition

If $X \subset \mathrm{Mod}(\mathcal{L})$ and $\bar{a} \in (\mathbb{N})^k$ then the Vaught transform $X^{*\bar{a}}$ is the set $\{\, M \mid \forall^* g \in S_\infty(\bar{a} \subset g \implies gM \in X) \,\}$.

Theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ is Borel and $k \in \mathbb{N}$ then there is a formula ϕ of $\mathcal{L}_{\omega_1\omega}$ with k free variables such that $M \in X^{*\bar{a}} \iff \phi^M(\bar{a})$.

López-Escobar's theorem follows from the special case when $k = 0$.

Outline of Vaught's proof of the stronger statement

Theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ is Borel and $k \in \mathbb{N}$ then there is a formula ϕ of $\mathcal{L}_{\omega_1\omega}$ with k free variables such that $M \in X^{*\bar{a}} \iff \phi^M(\bar{a})$.

Proof outline.

One shows that the class $\mathcal{K}$ of sets X satisfying the conclusion satisfies:

- $\mathcal{K}$ contains the subbasic open sets
- $\mathcal{K}$ is closed under complementation
- $\mathcal{K}$ is closed under countable intersections

These are shown by calculations using the elementary properties of the Vaught transforms. □

Metric structures

We seek analogs of the classical results above within the beautiful theory of metric structures and continuous logic.

Definition

A relational metric structure consists of:

- A complete metric space (M, d) of diameter 1
- Relations $R_i \colon M^{n_i} \to [0,1]$, each uniformly continuous (the modulus of continuity is specified in the language)

Motivation

The R_i are grey sets. If $R_i(\bar{a}) = 0$ then $\bar{a}$ is surely in R_i, and if $R_i(\bar{a}) > 0$ then its value measures the failure.

The space of separable metric structures

We will confine ourselves to metric structures whose underlying metric space is the Urysohn sphere $\mathbb{U}$, that is, the universal ultrahomogeneous separable metric space of diameter 1.

Definition

If $\mathcal{L}$ is a countable metric language with symbols R_i of arity n_i and modulus Δ_i, then we define the space of separable $\mathcal{L}$-structures

$$\mathrm{MMod}(\mathcal{L}) = \prod \mathrm{Unif}_{\Delta_i}(\mathbb{U}^{n_i}, [0,1]).$$

Here $\mathrm{Unif}_{\Delta}(X, Y)$ denotes the space of Δ-uniformly continuous functions from X to Y with the topology of pointwise convergence.

Remark

The Polish group $\mathrm{Iso}(\mathbb{U})$ of isometric bijections of $\mathbb{U}$ acts naturally on $\mathrm{MMod}(\mathcal{L})$, and its orbits are the isomorphism classes.

Universality of MMod($\mathcal{L}$)

As in the classical case, the space MMod($\mathcal{L}$) is a universal Iso($\mathbb{U}$)-space. In fact something stronger is true:

Proposition (Ivanov–Majcher-Iwanow)

Let $\mathcal{L}$ be a metric language with 1-Lipschitz symbols of arbitrarily large arity. Then for any Polish group G and Borel action $G \curvearrowright X$, there exists an equivariant embedding:

$$\alpha\colon G \leq \mathrm{Iso}(\mathbb{U}), \quad i\colon X \hookrightarrow \mathrm{MMod}(\mathcal{L})$$

such that i maps distinct orbits to distinct orbits.

Proof idea.

One uses the standard result of Uspenskij that Iso($\mathbb{U}$) is a universal Polish group, together with the methods of the Becker–Kechris result for the classical logic action. □

Continuous logic

Definition

Formulas of continuous logic (the logic for metric structures) are built from

- atomic formulas: the usual $R_i(x_1, \ldots, x_n)$, and the relation $d(x_1, x_2)$, which replaces equality
- connectives: for formulas $\phi_1, \ldots \phi_n$ we may form continuous combinations $f(\phi_1, \ldots, \phi_n)$
- quantifiers: for a formula ϕ we may form $\inf_x \phi(x, \ldots)$ and $\sup_x \phi(x, \ldots)$

The infinitary continuous logic $\mathcal{L}_{\omega_1\omega}$ additionally allows

- countable conjunctions and disjunctions: if ϕ_n are formulas we may form $\inf_n \phi_n$ and $\sup_n \phi_n$
  (Note: We require the moduli of uniform continuity of the ϕ_n be uniformly bounded.)

Main result: López-Escobar for metric structures

It is easy to see by induction on complexity that if ϕ is a sentence of $\mathcal{L}_{\omega_1\omega}$ then the evaluation map $M \mapsto \phi^M$ is Borel.
Ivanov–Majcher-Iwanow asked whether the converse holds.

Theorem (López-Escobar for MMod)

If $X\colon \mathrm{MMod}(\mathcal{L}) \to [0,1]$ is a Borel and isomorphism-invariant grey set, then there exists a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ such that for all $M \in \mathrm{MMod}(\mathcal{L})$ we have $X(M) = \phi^M$.

Remark

If X is $0,1$-valued we can additionally ensure that ϕ is $0,1$-valued.
It follows that if X is a genuine Borel and invariant subset of $\mathrm{MMod}(\mathcal{L})$ then X is axiomatized by a sentence of $\mathcal{L}_{\omega_1\omega}$.

A model-theoretic characterization of TVC

Corollary

The TVC is equivalent to the variant of Vaught's conjecture for classes axiomatized by a sentence of continuous $\mathcal{L}_{\omega_1\omega}$.

Proof.

(⇒) This holds because MMod(ϕ) is always Borel.

(⇐) Let X be a standard Borel G-space.

- There exists a language $\mathcal{L}$ and an equivariant embedding $i\colon X \hookrightarrow \mathrm{MMod}(\mathcal{L})$.
- By the main result, there exists a sentence ϕ of $\mathcal{L}_{\omega_1\omega}$ such that $i(X) = \mathrm{MMod}(\phi)$.
- By the VC for $\mathcal{L}_{\omega_1\omega}$, the image $i(X)$ has countably or perfectly many classes.
- Hence X has countably many or perfectly many orbits. □

Idea of the proof of the main result

As in Vaught's proof of López-Escobar's theorem, we wish to find a stronger version for grey sets which are not necessarily invariant.

Theorem

If $X \subset \mathrm{Mod}(\mathcal{L})$ is Borel and $k \in \mathbb{N}$ then there is a formula ϕ of $\mathcal{L}_{\omega_1\omega}$ with k free variables such that $M \in X^{*\bar{a}} \iff \phi^M(\bar{a})$.

Theorem

If $X \colon \mathrm{MMod}(\mathcal{L}) \to [0,1]$ is a Borel grey set and $k \in \mathbb{N}$, then there is a formula ϕ of continuous $\mathcal{L}_{\omega_1\omega}$ with k free variables such that for all $\bar{a} \in (\mathbb{U})^k$ we have $X^{*\bar{a}}(M) = \phi^M(\bar{a})$.

Definitions needed for the stronger statement

Theorem

If $X\colon \mathrm{MMod}(\mathcal{L}) \to [0,1]$ is a Borel grey set and $k \in \mathbb{N}$, then there is a formula ϕ of continuous $\mathcal{L}_{\omega_1\omega}$ with k free variables such that for all $\bar{a} \in (\mathbb{U})^k$ we have $X^{*\bar{a}}(M) = \phi^M(\bar{a})$.

Definition (Ivanov–Majcher-Iwanow)

- Let d_i enumerate a fixed dense subset of $\mathbb{U}$.
- basic grey sets of $\mathrm{Iso}(\mathbb{U})$: for $\bar{a} \in (\mathbb{U})^k$ let $[\bar{a}](g) = d(g^{-1}(d_1 \dots d_k), \bar{a})$.
- grey category quantifiers: For X a grey subset of $A \times B$ let $(\sup^*_{b \in B} X)(a) > r \iff \exists^* b \in B\ X(a,b) > r$.
- grey Vaught transforms: For X a grey subset of $\mathrm{MMod}(\mathcal{L})$ let $X^{*[\bar{a}]}(M) = \sup^*_{g \in G}(X(gM) - [\bar{a}](g))$.

Outline of the proof of the stronger statement

Theorem

If $X\colon \mathrm{MMod}(\mathcal{L}) \to [0,1]$ is a Borel grey set and $k \in \mathbb{N}$, then there is a formula ϕ of continuous $\mathcal{L}_{\omega_1\omega}$ with k free variables such that for all $\bar{a} \in (\mathbb{U})^k$ we have $X^{*\bar{a}}(M) = \phi^M(\bar{a})$.

Proof outline.

One again shows that the class $\mathcal{K}$ of grey sets X satisfying the conclusion satisfies:

- $\mathcal{K}$ contains sufficiently many "basic" grey sets
- $\mathcal{K}$ is closed under negation $r - X$
- $\mathcal{K}$ is closed under linear combinations
- $\mathcal{K}$ is closed under $\sup_n$ and $\inf_n$

To conclude that $\mathcal{K}$ must contain all Borel functions, one argues as in the Lebesgue–Hausdorff theorem for Baire class functions. □