The complexity of classification problems

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April 2016

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Borel complexity theory

Classification problems in mathematics

What is classification?

Classification means explicitly finding invariants that completely determine the objects up to equivalence.

- The objects are presented in a concrete way. (A group by its multiplication table, a graph by its incidence relation, etc.)
- The equivalence is an equivalence relation on the space of objects. (Isomorphism, isometry, conjugacy, etc.)
- Explicit means the mapping from object to invariant is reasonably explicit. (We use Borel definability.)

Remark

The approach dates to the 1990s and has gained prominence thanks in part to an number of striking applications.

Formal definition

Definition

If E is an equivalence relation on a standard Borel space X, we say E is completely classifiable if there is a Borel function $f : X \to \mathbb{R}$ satisfying

$$x\, E\, x' \iff f(x) = f(x')$$

Example

The finitely generated abelian groups are classified up to isomorphism by a code for the sequence of cyclic factors in the primary decomposition.

Example

The countable divisible groups are classified up to isomorphism by $f(G) =$ the sequence which gives the number of copies of $\mathbb{Q}$ and $\mathbb{Z}/p^{\infty}\mathbb{Z}$ for all p.

Relative complexity

Definition

If E, F are equivalence relations on standard Borel spaces X, Y, we say E is Borel reducible to F (written $E \leq_B F$) if there is a Borel function $f\colon X \to Y$ satisfying

$$x \, E \, x' \iff f(x) \, F \, f(x')$$

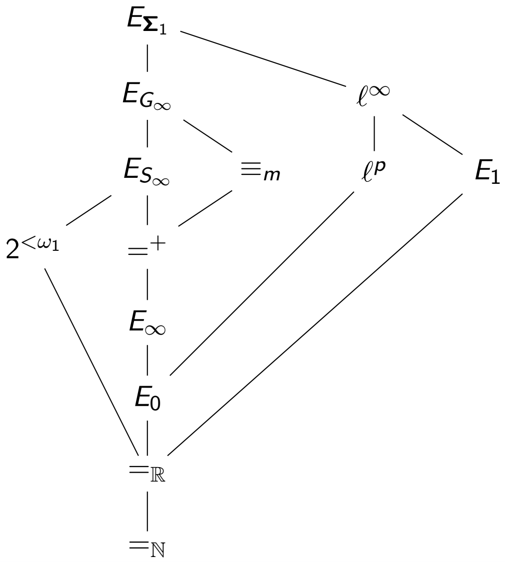
Motivation

We interpret $E \leq_B F$ as: the classification up to E-equivalence is no harder than the classification up to F-equivalence.

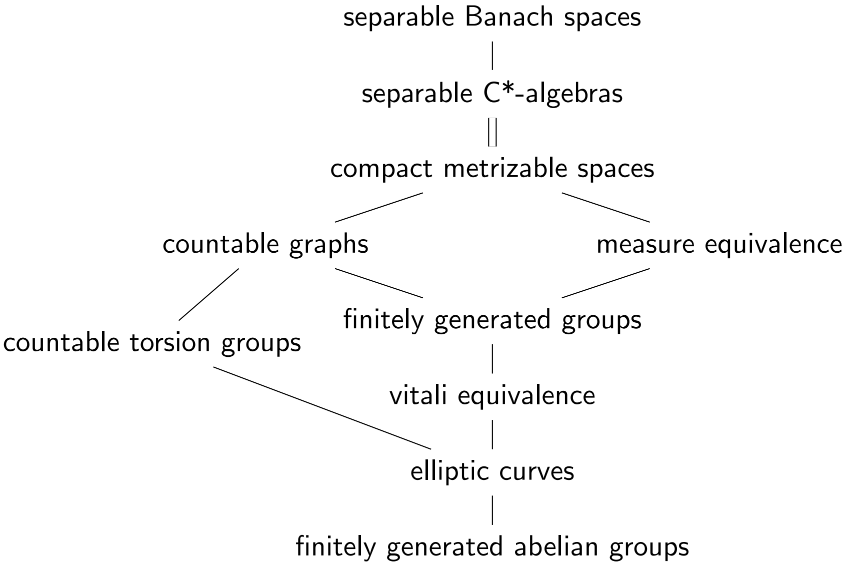
Remark

It is necessary that we impose a definability constraint on the reduction functions. The slogan is that Borel = Explicit.

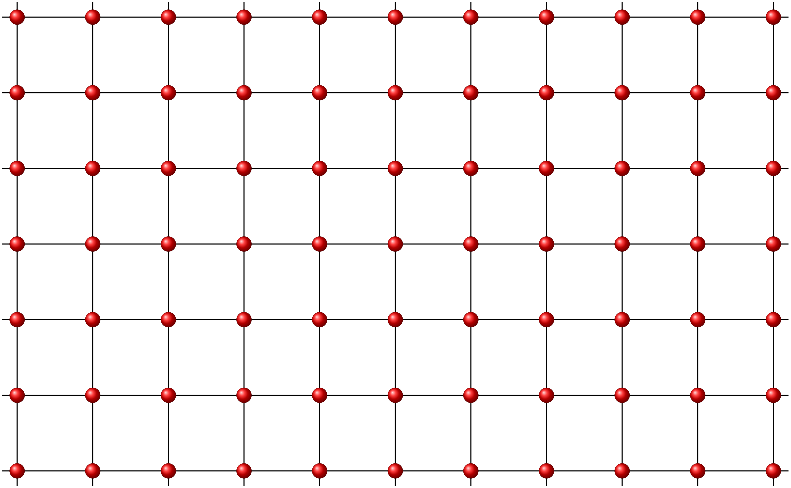
A map of benchmark complexities



A map of naturally occurring classifications



Countable groups



Torsion-free abelian groups

Definition

A group A is torsion-free abelian if it is isomorphic to a subgroup of some power $\mathbb{Q}^n$.

The least possible such n is called the rank of A (it may be infinite).

Theorem (Baer, 1937)

The torsion-free abelian groups of rank 1 *are classified by the sequence of p-heights of one of its elements (up to a finite error).*

Remark

In our setting, this can be viewed as an E_0 classification.

Chain

Kurosh (1937) and Mal'cev (1938) independently generalized Baer's work to higher ranks. But the classification was regarded as unsatisfactory, because the invariants were themselves complicated.

Question

Is there a satisfactory classification of torsion-free abelian groups of finite rank?

Theorem (Hjorth 1998, Thomas 2002)

The classification problem for torsion-free abelian groups increases strictly in Borel complexity with the rank:

$$R_1 <_B R_2 <_B R_3 <_B \cdots$$

Antichain

Within a fixed rank, it is possible to categorize the torsion-free abelian groups by their divisibility properties.

Definition

A torsion-free abelian group is said to be p-local if it is (infinitely) divisible by every prime $q \neq p$.

Theorem (Thomas, 2003)

If p, q are distinct primes, then the classification problems for p-local and q-local torsion-free abelian groups of fixed finite rank are incomparable:

$$R_{n,p} \perp_B R_{n,q}$$

Quasi-isomorphism

It is also natural to work with the following coarser classification.

Definition

Torsion-free abelian groups A, B are quasi-isomorphic if they are isomorphic up to a finite index error (there exist finite-index subgroups A', B' of A, B such that $A' \cong B'$.)

Motivation

- While torsion-free abelian groups need not admit a unique decomposition into irreducible groups, they always have a quasi-unique quasi-decomposition.
- The quasi-isomorphism relation arises in an auxiliary fashion in the work of Kurosh, Mal'cev, and of Hjorth, Thomas.

Incomparable problems

Question

Is it simpler to classify the torsion-free abelian groups up to isomorphism or up to quasi-isomorphism?

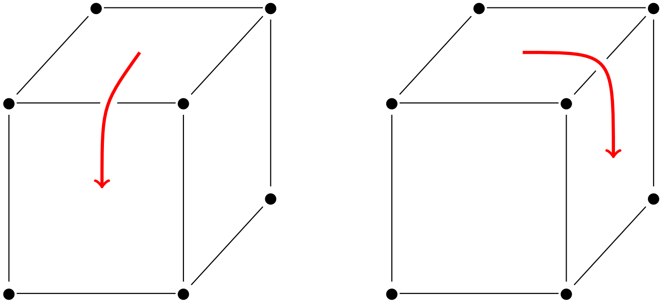
Theorem (C–)

The isomorphism and quasi-isomorphism problems on torsion-free abelian groups of finite rank are incomparable with respect to Borel reducibility.

Remark

This was perhaps the first incomparability result, within this level of the hierarchy, when no obvious invariant (such as the prime p) is available to distinguish the two relations.

Automorphisms



The conjugacy problem for automorphisms

Definition

Let M be any countable structure (group, graph, etc). Automorphisms $\phi, \psi \in \mathrm{Aut}(M)$ are conjugate if there exists $\alpha \in \mathrm{Aut}(M)$ such that $\psi = \alpha\phi\alpha^{-1}$.

Motivation

Classifying automorphisms up to conjugacy is analogous to writing the class equation in the case of a finite structure.

Example

The cube has five conjugacy classes: identity, 90° rotations, 120° rotations, 180° rotations about a face, and 180° rotations about an edge (and several more if reflections are allowed).

Homogeneous structures

We will investigate the conjugacy problem for structures with very rich automorphism groups.

Definition

A structure A is called homogeneous if for every finite subset $A_0 \subset A$ and embedding $f_0 \colon A_0 \to A$, the f_0 lifts to an automorphism $f \in \mathrm{Aut}(A)$.

Examples

- The rational order $(\mathbb{Q}, <)$ is homogeneous; here one can use a back-and-forth construction to lift f_0.
- The integer order $(\mathbb{Z}, <)$ is not homogeneous; consider f_0 mapping the pair $1, 3$ to the pair $2, 3$.

Homogeneous graphs

Lachlan and Woodrow (1980) classified the countable homogeneous graphs.

Theorem (C–, Ellis)

If G is a countable homogeneous graph, then the conjugacy problem for $\mathrm{Aut}(G)$ *is either:*

- $=_{\mathbb{R}}$ *(for graphs $m \cdot K_\infty$ and $\infty \cdot K_n$);*
- $=^+$ *(for the graph $\infty \cdot K_\infty$); or*
- E_{S_∞} *(for the random graph Γ and random K_n-free graph Γ_n).*

Homogeneous digraphs

Cherlin (1998) classified the countable homogeneous digraphs.

Theorem (C–, Ellis)

If G is a homogeneous digraph, then the conjugacy problem for $\mathrm{Aut}(G)$ *is either:*

- $=_{\mathbb{R}}$ *(for digraphs I_∞, $\infty \cdot C_3$, $C_3[\infty]$); or*
- E_{S_∞} *(for digraphs $n \cdot \mathbb{Q}$, $\mathbb{Q}[n]$, $\hat{\mathbb{Q}}$, $n \cdot S(2)$, $S(2)[n]$, $S(3)$, $\mathcal{P}$, $\mathcal{P}(3)$, $n \cdot \mathcal{T}$, $\mathcal{T}[n]$, $\hat{\mathcal{T}}$, Γ_n, Λ_n, $\Gamma_{\mathcal{F}}$, $n * K_\infty$, $\infty \hat{*} K_\infty$).*

Remark

In recent work, we give abstract conditions that guarantee $\mathrm{Aut}(G)$ has complexity E_{S_∞}, reducing tedious verification and providing new examples.

Trees

Recent investigation includes regularly branching trees (connected acyclic graphs).

These are not fully homogeneous, but they are 1-homogeneous (vertex-transitive).

Theorem (Beserra)

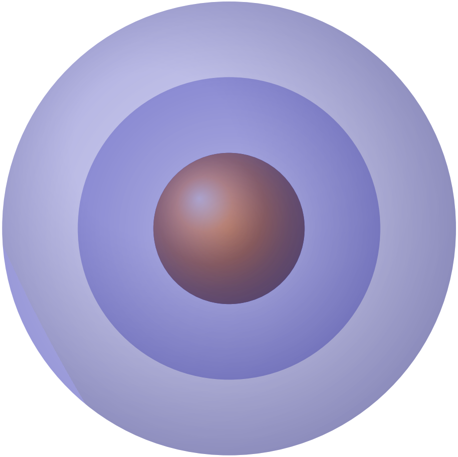
If T is a countable regularly branching tree, then the classification of $\mathrm{Aut}(T)$ up to conjugacy is either:

- $=_{\mathbb{N}}$ *(for the 2-regular tree $\mathbb{Z}$);*
- E_∞ *(for the n-regular tree, $n > 2$); or*
- E_{S_∞} *(for the ∞-regular tree).*

Question

What complexities arise from non-regularly branching trees?

Metric structures



C*-algebras

Definition

If X is a locally compact Hausdorff space, its corresponding commutative C*-algebra is the ring $C_0(X)$ = the continuous complex valued functions on X vanishing at ∞.

Fact

Every commutative C-algebra is isomorphic to a closed subspace of the diagonal of $B(\mathcal{H})$ = the bounded operators on some Hilbert space $\mathcal{H}$.*

Definition

A general C*-algebra is any norm-closed, *-subalgebra of $B(\mathcal{H})$.

Remark

The C*-algebras generalize topological spaces in a sense, and play a role in representation theory, quantum mechanics, and more.

Classification of C*-algebras

The classification of C*-algebras is an established research area (Elliott, Effros, and many others). Recently, the classification has been studied through the lens of Borel complexity theory.

Proposition (Farah, Toms, Törnquist 2012)

The separable C-algebras can be parameterized by a countable sequence of elements of $B(\mathcal{H})$ which is dense in the algebra.*

Theorem (Elliott, Farah, et al 2013, Sabok 2015)

The classification of separable C-algebras is E_{G_∞} (complete for isometric classifications).*

Operator systems

The scope of the classification can be extended to include the wider class of operator systems: norm-closed, *-closed, unital, vector subspaces of $B(\mathcal{H})$.

The operator systems, together with their completely positive mappings, are a focus of current research in functional analysis, and indeed many classical problems are best addressed in this framework.

The proofs of the results on the previous slide can be used to give:

Theorem

The classification of separable operator systems, up to complete isometric isomorphism, is also E_{G_∞}.

Finite-dimensional operator systems

The finite-dimensional C*-algebras are easily understood; they are direct sums of full matrix algebras.

Arveson (2010) classified the finite-dimensional operator systems that can be embedded in a finite-dimensional C*-algebra.

The general classification was believed to be very complex by several practitioners.

Theorem (Argerami, C–, et al)

The classification of finite dimensional operator systems, up to complete isometric isomorphism, is $=_{\mathbb{R}}$ *(completely classifiable).*

Remark

This result is one of several positive applications of Borel complexity theory; many results show a problem is intractable.

Conclusion

Borel complexity theory:

- Clarifies historical and ongoing classification programs
- Provides a fresh source for new questions about classical theories
- Interacts nontrivially with algebra, functional analysis, dynamical systems, geometric group theory, and of course set theory and model theory
- Has a rich theory all its own, with many things known and many very challenging open problems.

Thank you!