Infinite-time Turing machines and Borel reducibility

Samuel Coskey

The City University of New York

In this document I will outline a couple of recent developments, due to Joel Hamkins, Philip Welch and myself, in the theory of infinite-time Turing machines. These results were obtained with the idea of extending the scope of the study of Borel equivalence relations, an area of descriptive set theory. I will introduce the most basic aspects of Borel equivalence relations, and show how infinite-time computation may provide insight into this area.

1 Infinite-time Turing machines

We begin by describing a model of transfinite computation called the infinite-time Turing machine (ITTM), invented by Hamkins and Kidder and introduced by Hamkins and Lewis [1]. Like an ordinary Turing machine, an ITTM is a finite state machine with an input tape, an output tape, and a read/write head. The key addition is that when an ITTM reaches a limit ordinal numbered step, the program enters a special limit state, the read/write head is reset to the left, and the value of each cell is set to the limit of the previous values in that cell (or else zero, if the value has flipped unboundedly many times).

Definition 1. We say that a partial function f is *ittm-computable* iff there exists an ITTM which on input x, writes $f(x)$ if it is defined and diverges otherwise. We say that f is *eventually computable* iff there exists an ITTM such that on input x, the output tape eventually converges to $f(x)$ (it need not halt) if it is defined and diverges otherwise.

It is also possible to allow an ITTM an *oracle tape*, and doing so allows one to relativize computations to a real parameter. For $x, y \in 2^{\mathbb{N}}$, we say that x is ittm-computable from y, written $x \leq_\infty y$, iff there exists an ITTM with oracle y which writes x. Letting $x \equiv_\infty y$ iff $x \leq_\infty y$ and $y \leq_\infty x$, we say that $[x]_{\equiv_\infty}$ is the *ittm-degree* of x. The *eventual degree* relation $\equiv_{e\infty}$ is defined analogously.

The ittm-degrees share many of the most basic properties of the classical Turing degrees. For instance, since there are still only countably many ittm programs, there are continuum many ittm-degrees. It is natural to ask for the descriptive set-theoretic strengthening, namely that there exists a *perfect set* of ittm-degrees. This was recently established by Welch, building upon his earlier work [2] which had shown that there are continuum many minimal ittm-degrees.

Theorem 1 (Welch). *There exists a perfect set of pairwise ittm-computably incomparable reals. There exists a perfect set of pairwise eventually computably incomparable reals.*

We next describe the pointclasses of decidable sets that correspond to these new notions of computation. We say that a set is of class D or *ittm-decidable* iff its characteristic function is ittm-computable, and of class sD or *semi ittm-decidable* iff it is the domain of a ittm-computable function. The eventually ittm-decidable and semi ittm-decidable classes E, sE are defined analogously. These classes are all provably Δ^1_2, and fit into the usual projective hierarchy as follows.

$$\begin{array}{ccccccccccc} \Sigma^1_1 & & & & sD & & & & sE & & \\ & \subset & D & \subset & & \subset & E & \subset & & \subset & \Delta^1_2 \\ \Pi^1_1 & \subset & & \subset & sD' & \subset & & \subset & sE' & \subset & \end{array}$$

In fact, each of these new pointclasses is contained in the class $\text{prov}(\Delta^1_2)$ of ZFC provably Δ^1_2 sets. It follows that the ittm-decidable sets and the ittm-computable functions are Lebesgue measurable, a fact which will be important later on.

2 Equivalence relations

In this section, we investigate some of the advantages of using notions from infinite-time computability in the study of Borel equivalence relations. This is the second area of logic which has received the ITTM treatment, the first being computable model theory [3].

The theory of Borel equivalence relations revolves around the following complexity notion. If E, F are equivalence relations on $2^{\mathbb{N}}$, then following [4] and [5] we say that E is *Borel reducible* to F, written $E \leq_B F$, iff there exists a Borel function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

$$x \; E \; y \iff f(x) \; F \; f(y)$$

for all $x, y \in 2^{\mathbb{N}}$. The function f is said to be a Borel *reduction* from E to F.

For applications, we think of the equivalence relations E, F as representing classification problems from some other area of mathematics. For instance, since any group with domain $\mathbb{N}$ is determined by its multiplication function, studying the classification problem for countable groups amounts to studying the isomorphism equivalence relation on a suitable subspace of $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$. For many more examples, see [**?**, Section 1.2].

Now, recall that the Borel sets correspond to the bold-face pointclass $\boldsymbol{\Delta}^1_1$. Since we wish to generalize the Borel reductions, we shall use from now on the following bold-face analog of the ittm-computable functions. Namely, we say that f is *bold-face ittm-computable* iff there exists a $z \in 2^{\mathbb{N}}$ such that f is computed by an ITTM with oracle z. The *bold-face eventually computable* functions are defined analogously.

Definition 2. We say that E is *ittm-computably reducible* to F, written $E \leq_c F$, iff there is a (bold-face) ittm-computable reduction from E to F. We say that E is *eventually reducible* to F, written $E \leq_e F$, iff there is a (bold-face) eventually computable reduction from E to F.

A key observation in the study of ittm-computable and eventual reducibility is they hardly differ from Borel reducibility on many well-studied equivalence relations. Indeed, it is still unknown whether there exist Borel equivalence relations E, F such that $E \leq_c F$ but $E \not\leq_B F$. On the other hand, there exist natural equivalence relations which are so complex that Borel reducibility does not capture their relationship, and computable reducibility does. Let $x \equiv_{\mathsf{CK}} y$ iff x, y compute (in the ordinary sense) the same countable ordinals and $x \cong_{\mathsf{WO}} y$ iff x and y code isomorphic well-orders. Then we have the following:

Theorem 2 (Hamkins-Coskey). *The equivalence relations $\cong_{\mathsf{WO}}$ and $\equiv_{\mathsf{CK}}$ are Borel incomparable but ittm-computably bireducible.*

We are presently motivated by the following two goals:

– Explore the extent of the analogy between $\leq_B$ and either $\leq_c$ or $\leq_e$.
– Use the notion of ittm-computable complexity to analyze interesting relations of high complexity, where Borel reducibility is not appropriate.

3 Countable equivalence relations

Some progress towards these goals has been made in the countable case. An equivalence relation E is said to be *countable* iff every E-class is countable. The collection of countable Borel equivalence relations has undergone a great deal of study; let us outline the basic complexity picture. The least complex countable Borel equivalence relation is the equality relation $=$. Its immediate successor is the *almost equality* relation E_0, defined by $x \; E_0 \; y$ iff $x_n = y_n$ for all but finitely many n. There is also a most complex countable Borel equivalence relation, called E_∞. The remaining ones lie in the interval (E_0, E_∞), and it has been shown by Adams and Kechris [6] that every Borel partial ordering embeds into $\leq_B$ on this interval.

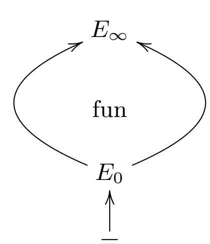

Fig. 1. The countable Borel equivalence relations

We wish to obtain all of the same conclusions for the countable ittm-computable equivalence relations, but it is not clear that this will be possible. Instead, we must work with the following infinite-time generalizations of the countable Borel equivalence relations.

Definition 3. The equivalence relation E is *write-outable* iff there exists a (bold-face) ittm-computable function such that for all x, $f(x)$ codes an enumeration of $[x]_E$. The *eventually write-outable* equivalence relations are defined analogously.

For instance, every countable Borel equivalence relation is write-outable, and the ittm-degree equivalence relation $\equiv_\infty$ is eventually write-outable but not write-outable. The write-outable equivalence relations share many properties with the countable Borel equivalence relations. The first result is the following, which essentially an immediate consequence of Theorem 1.

Corollary 1. *The equality relation $=$ is computably reducible to every write-outable equivalence relation.*

Secondly, it follows from a classical argument (from the Borel case) that E_∞ is universal for the write-outable equivalence relations. Lastly, since the arguments of Adams and Kechris are measure-theoretic, and every ittm-computable function is measurable, we also have that any Borel partial order embeds into the write-outable equivalence relations. Moreover, the analog of each of these results holds for the eventually write-outable relations under eventual reducibility. It remains only to show that E_0 is the unique immediate successor of $=$, in the sense of ittm-computable or eventually computable reducibility.

We close by mentioning an open problem in countable Borel equivalence relations. The ordinary Turing degree relation $\equiv_T$ is Borel reducible to E_∞, but it is unknown whether $\equiv_T$ is bireducible with E_∞. It is known, however, that if $\equiv_T$ is indeed universal then the Martin Conjecture must fail. Since the ittm-degree relation $\equiv_\infty$ is eventually write-outable, it is eventually reducible to E_∞, but it remains unknown whether $\equiv_\infty$ is ittm-bireducible with E_∞.

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