TWO ARTILCES ON THE CLASSIFICATION OF SEPARABLE C*-ALGEBRAS

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G. A. Elliott, I. Farah, V. I. Paulsen, C. Rosendal, A. S. Toms, and A. Törnquist. **The isomorphism relation for separable C*-algebras**. *Mathematics research letters*. vol. 20 no. 6 (2013), pp. 1071–1080.

Marcin Sabok. **Completeness of the isomorphism problem for separable C*-algebras**. *Inventiones mathematicae*. Preprint available at arXiv:1306.1049.

The two articles together identify precisely the complexity of the classification problem for separable C*-algebras. The first paper establishes that C*-algebras are classifiable by the orbits of a Polish group action, and the second paper shows that this problem is of maximum complexity among all such problems. In the terminology of invariant descriptive set theory that we will explain shortly, we say that the classification of separable C*-algebras is *complete orbitable*. This is indeed very complex, but not as complex as some problems have been found to be. For example it is the same complexity as the isometric classification of separable Banach spaces, but not as complex as the homeomorphic classification of separable Banach spaces.

Invariant descriptive set theory (also called Borel equivalence relations) is the rigorous study of the complexity of classification problems in mathematics. In this approach, the objects to be classified are parameterized by elements of a standard Borel space, and the classification problem itself is identified as an equivalence relation on that space. For example, since any group with underlying set $\mathbb{N}$ is determined by its three-place multiplication relation, the countable groups can be parameterized by elements of $\mathcal{P}(\mathbb{N}^3)$. Though it is a little more complicated, Farah–Toms–Törnquist recently gave a natural parameterization of the separable C*-algebras.

Classification in C*-algebras has a long and illustrious history. For example, the well-known Elliott program and its extensions seek to classify C*-algebras by their K-theoretic invariants. Traditionally, analysts have sought category-theoretic classifications for C*-algebras. However, recently the descriptive set-theoretic point of view has supplemented and clarified the categorical ones.

The key notion in invariant descriptive set theory is that of *Borel reducibility*. Here, if E, F are equivalence relations on standard Borel spaces X, Y, then E is Borel reducible to F if there exists a Borel function $f\colon X \to Y$ such that $x \; E \; x' \iff f(x) \; F \; f(x')$. In other words, the F-classes

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may be used as complete invariants for the classification problem up to E. The condition that the reduction function f be Borel amounts to the requirement that the invariants are *explicitly definable*

With this notion in hand, classification problems can be compared against one another and placed within a hierarchy. Although the hierarchy is wild, many natural problems fall into one of a few well-studied layers. To begin, the lowest level in the hierarchy consists of the relations that are *completely classifiable*, that is, Borel bireducible with the equality relation on some (standard Borel) space of complete invariants. For example, countable divisible groups are completely classified by the sequence which lists for each prime p the number of copies of $\mathbb{Z}(p^\infty)$ that occur as a factor. Similarly, Glimm's classification of UHF C*-algebras implies they are classified by the sequence which lists for each prime p the maximum n such that $M_{p^n}(\mathbb{C})$ unitally embeds.

A moderately higher layer consists of those equivalence relations which are *classifiable by countable structures*, that is, Borel reducible to the isomorphism relation on the class of countable first-order structures (in a countable language). Elliott's well-known classification implies that AF algebras are classifiable by countable structures, and a result of Camerlo–Gao implies that it is *complete* among such relations, that is, isomorphism of countable structures is itself Borel reducible to isomorphism of AF algebras.

The broadest class of relations that are still considered to be in some sense classifiable consists of those that are orbitable. Before defining it, we note that an equivalence relation is classifiable by countable structures if and only if it is Borel reducible to the orbit equivalence relation of some action of the Polish group S_∞. More broadly then, an equivalence relation is called *orbitable* if it is Borel reducible to the orbit equivalence relation of some action of some Polish group. Once again we call such a relation *complete orbitable* if all orbitable relations are themselves reducible to it. For example, Clemens and Gao–Kechris have shown that the isometric classification of Polish metric spaces is complete orbitable.

The article of Elliott et al. establishes that the classification of separable C*-algebras is orbitable. The proof turns out to be a straightforward application of the basic theory of *model theory for metric structures*. Without going into detail, model theory for metric structures is a proper generalization of first order model theory where discrete (not topologized) domains are replaced by bounded metric spaces, and the Boolean truth values are replaced by the interval $[0, 1]$. In this model theory, together with its corresponding logic, it is possible to realize a C*-algebra as a structure.

The key observation of Elliott et al. is the following: just as the isomorphism relation on any class of countable first-order structures is given by an action of S_∞, the isometry relation on any class of separable metric structures is reducible to an action of a universal Polish group G_∞. The particular instance of G_∞ used here is the isometry group of the universal Urysohn metric space $\mathbb{U}$.

The intuition behind this statement is so simple that in hindsight it is surprising the problem had circulated for several years without being solved (it seems it was a matter of getting the right people together in the same room). Still there are several technical details that needed to be verified in the article. First, one must check that metric structures can be transferred to a subset of $\mathbb{U}$ in such a way that isometric isomorphisms are witnessed by isometries of all of $\mathbb{U}$. This is done using an adaptation of the Katětov construction of $\mathbb{U}$ together with several selection results from descriptive set theory (including a new one in the Appendix). And second one needs to verify that the natural parameterization of separable C*-algebras as metric structures is equivalent to the parameterization of C*-algebras used previously.

We now turn to Sabok's article, which establishes the complementary fact that the classification of separable C*-algebras is complete orbitable. In fact, it shows that the isometric classification of Polish metric spaces is Borel reducible to the affine homeomorphic classification of Choquet simplices. Here a *Choquet simplex* (or just *simplex*) is the infinite-dimensional generalization of an ordinary simplex: a compact convex set with the additional property that each point is the barycenter of a unique measure on the extreme boundary. The main result follows from this reducibility result, since the affine classification of simplices is known to be Borel reducible to isomorphism of C*-algebras. In fact it is Borel reducible to the class of AI algebras—limits of matrix algebras over copies of $C[0,1]$.

Sabok's proof consists of several steps, which ultimately show how to construct a Borel reduction f from Polish metric spaces to simplices. For f to be a Borel reduction f must map isometric spaces to affinely homeomorphic spaces, and moreover the value $f(X)$ must "remember" (or encode) the metric structure of X. Since the construction consists of taking several inverse limits, the proof that $f(X)$ is independent of the isometric copy of X relies on the repeated use of intertwining arguments.

For the construction, Sabok begins by showing how to realize a given Polish metric space X as a dense G_δ subset of the extreme boundary of some compact convex subset $S(X)$ of the Hilbert cube. The definition of $S(X)$ is made relative to an embedding of X into the Urysohn space $\mathbb{U}$. The key lemma states that if X is embedded as a very special, thin subset of $\mathbb{U}$, then the corresponding $S(X)$ will in fact be a simplex. The proof that any space X admits such a special embedding into $\mathbb{U}$ relies on yet another clever modification of the Katětov construction of the Urysohn space.

In the next steps, Sabok extends $S(X)$ to a larger simplex $f(X)$ by adding a pattern of new cone points that encode the original metric structure on X. Note that the metric on X can be conveyed by a countable amount of data: the digits of $d(x,y)$ for x,y lying in a countable dense subset $D \subset X$. To encode this data, one first makes room by adding countably many new cone points $c_{n,i,x,y}(x), c_{n,i,x,y}(y)$ corresponding to each pair $x,y \in D$. Finally one codes whether the n^{th} digit of $d(x,y)$ is equal to i by "marking" certain points along the segment between $c_{n,i,x,y}(x), c_{n,i,x,y}(y)$ with further new cone points.

Since Sabok only uses Polish metric spaces X that are perfect, the cone points used in the coding can be distinguished from the extreme boundary of the original simplex $S(X)$ as the isolated extreme boundary points of $f(X)$. The final technical lemma then verifies that the digits of $d(x,y)$ can be recovered by analyzing which cone points are limits of isolated extreme boundary points.

We conclude by mentioning a recently announced and closely related theorem of J. Zielinski, which states that the homeomorphic classification of compact metric spaces is complete orbitable. This theorem strengthens Sabok's result, since it has long been known that homeomorphism of compact metric spaces is Borel reducible to equivalence of Choquet simplices. Meanwhile since homeomorphism of compact metric spaces is bireducible with isomorphism of *commutative* C*-algebras, it follows that this latter class of C*-algebras is also complete orbitable. Zielinksi's proof uses Sabok's result as a black box, and shows that equivalence of Choquet simplices is Borel reducible to homeomorphism of compact metric spaces.