Stably Free Modules over the Klein Bottle

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Polynomials

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Then the collection of all polynomials with coefficients from R is a ring.

Denote this ring as

$$R[x].$$

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Let R be a ring.

A **Laurent polynomial** is a polynomial which allows negative exponents.

Then the collection of all Laurent polynomials with coefficients from R is a ring.

Denote this ring as

$$R[x, x^{-1}].$$

Polynomials

Let R be a ring and let $\sigma : R \longrightarrow R$ be a ring isomorphism.

A **skew Laurent polynomial** is a Laurent polynomial but with the extra condition that $rx = xr^{\sigma}$.

Then the collection of all skew Laurent polynomials with coefficients from R is a ring.

Denote this ring as

$$R[x, x^{-1}; \sigma].$$

Projective Modules

Definition - Free Module

An R-module M is **free** iff $M \cong \bigoplus_n R$ for some cardinal n.

$\{$Free Modules$\}$

Projective Modules

Definition - Projective Module

An R-module P is **projective** iff P is a direct summand of a free module, that is, $\exists Q$ R-module such that $P \oplus Q \cong R^n$.

$$\{\text{Free Modules}\} \subseteq \{\text{Projective Modules}\}$$

Projective Modules

Definition - Stably Free Module

An R-module P is **stably free** iff there exists natural numbers m, n such that $P \oplus R^m \cong R^n$.

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Projective modules over group algebras play a key role in many aspects of geometry and topology.

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Theorem (The Quillen-Suslin Theorem (1976))

Let k be a commutative ring. Then all projective modules over $k[x_1, \ldots, x_n]$ *are free.*

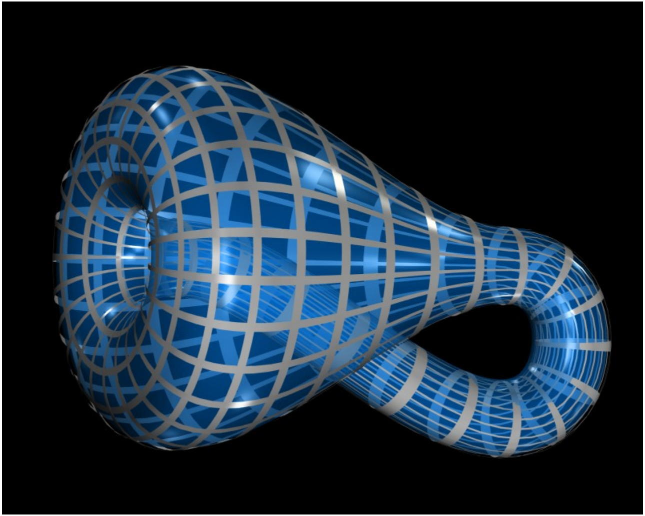
Quillen-Suslin Theorem

Projective modules over group algebras play a key role in many aspects of geometry and topology.

Theorem (Generalized Quillen-Suslin Theorem (Swan 1978))

Let k be a commutative ring. Then all projective modules over $k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ are free.

This is a Klein Bottle



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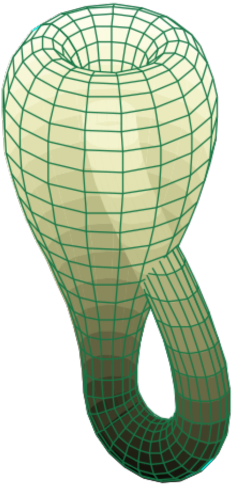

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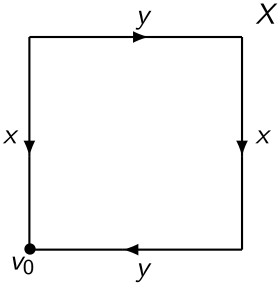

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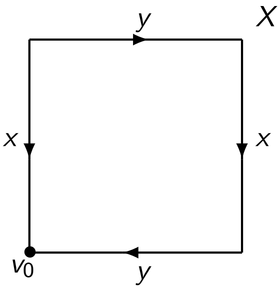

We see from the fundamental square, that

$$\begin{aligned}\pi(X, v_0) &= \langle x, y \mid x^{-1}yx = y^{-1}\rangle \\ &= \langle x, y \mid yx = xy^{-1}\rangle\end{aligned}$$

Call this group G.

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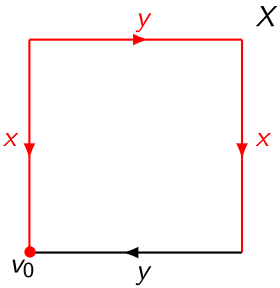

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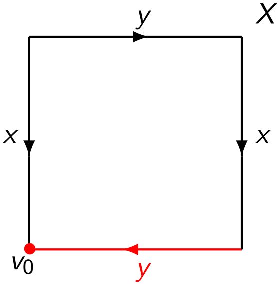

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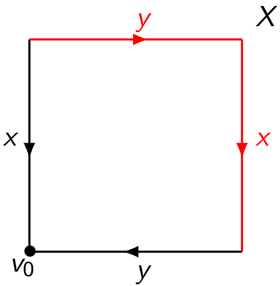

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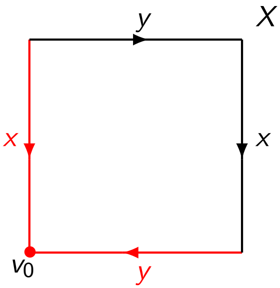

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The Ring $\mathbb{Z}G$

In light of the presentation of G,

$$\mathbb{Z}G = R[x, x^{-1}; \sigma]$$

where

$$R = \mathbb{Z}[y, y^{-1}]$$

and σ is the isomorphism induced by

$$\sigma : y \longmapsto y^{-1}.$$

In particular,

$$yx = xy^{-1}$$

as polynomials in $\mathbb{Z}G$.

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$$\langle y, x^2 \rangle \cong \mathbb{Z} \times \mathbb{Z},$$

$$\big(G : \langle y, x^2 \rangle\big) = 2.$$

This tiny bit of noncommutativity guarantees the existence of nonfree projective modules over the Klein bottle.

The $\mathbb{Z}G$-module K

Let

$$r = 1 + y + y^3 \in \mathbb{Z}G,$$

and let

$$s = r^{\sigma^{-1}} \in \mathbb{Z}G.$$

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$$\begin{aligned} K &= \{f \in \mathbb{Z}G \mid rf = (x+s)g,\ g \in \mathbb{Z}G\} \\ &\cong \langle r \rangle \cap \langle x+s \rangle \\ &\cong \langle 1 + y + y^3 \rangle \cap \langle x + 1 + y^{-1} + y^{-3} \rangle \end{aligned}$$

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Some Important Maps

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$$i : f \mapsto \begin{pmatrix} f \\ -\theta(f) \end{pmatrix}.$$

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Let $\pi : \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G$ as

$$\pi = \begin{pmatrix} r & x + s \end{pmatrix}.$$

K is Stably Free

Theorem

The sequence $0 \longrightarrow K \xrightarrow{i} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\pi} \mathbb{Z}G \longrightarrow 0$ *is exact.*

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$$\begin{aligned} \pi\left(\begin{array}{c} (sx^{-2}) \\ (x^{-1} - rx^{-2}) \end{array}\right) &= r(sx^{-2}) + (x+s)(x^{-1} - rx^{-2}) \\ &= r(r^{\sigma^{-1}}x^{-2}) + (x + r^{\sigma^{-1}})(x^{-1} - rx^{-2}) \\ &= 1. \end{aligned}$$

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& \pi \begin{pmatrix} f \\ g \end{pmatrix} = 0 \\
\Rightarrow \quad & rf + (x+s)g = 0 \\
\Rightarrow \quad & rf = -(x+s)g = (x+s)(-g) \\
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The sequence $0 \longrightarrow K \xrightarrow{i} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\pi} \mathbb{Z}G \longrightarrow 0$ *is exact.*

Now, the above sequence is exact. But $\mathbb{Z}G$ is projective. Thus, $\exists \mathscr{s} : \mathbb{Z}G \to \mathbb{Z}G^2$ such that

$$\pi \mathscr{s} = \mathbf{1}_{\mathbb{Z}G}.$$

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The sequence $0 \longrightarrow K \xrightarrow{\ i\ } \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\ \pi\ } \mathbb{Z}G \longrightarrow 0$ *is exact.*

Define $\mathscr{s} : \mathbb{Z}G \to \mathbb{Z}G \oplus \mathbb{Z}G$ as

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Corollary

K is a stably free $\mathbb{Z}G$-module, that is, $K \oplus \mathbb{Z}G \cong \mathbb{Z}G \oplus \mathbb{Z}G$.

Presentation of K

Define $p : \mathbb{Z}G \oplus \mathbb{Z}G \longrightarrow K$ as

$$p : \vec{m} \mapsto i^{-1}(\vec{m} - \mathscr{s}\pi\vec{m}).$$

Presentation of K

Define $p : \mathbb{Z}G \oplus \mathbb{Z}G \to K$ as

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Theorem

The sequence $0 \longrightarrow \mathbb{Z}G \xrightarrow{\mathscr{s}} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$ *is exact.*

Presentation of K

Define $p : \mathbb{Z}G \oplus \mathbb{Z}G \to K$ as

$$p = \begin{pmatrix} 1 - (sx^{-2})r & -(sx^{-2})(x+s) \end{pmatrix}.$$

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Hence, we see the following diagram with exactness in both directions.

$$\begin{tikzcd}
0 \arrow[r, leftrightarrow] & K \arrow[r, bend left, "i"] & \mathbb{Z}G^2 \arrow[l, bend left, "p"] \arrow[r, bend left, "\pi"] & \mathbb{Z}G \arrow[l, bend left, "\mathscr{s}"] \arrow[r, leftrightarrow] & 0
\end{tikzcd}$$

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Generators of K

$p : \mathbb{Z}G^2 \longrightarrow K$ is a surjective map.

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$$K = \Big\langle x^2 - y^3 - y^2 - y - 3 - y^{-1} - y^{-2} - y^{-3}, \\ xy^3 + xy + x + 1 + 2y^{-1} + y^{-2} + 2y^{-3} + 2y^{-4} + y^{-6} \Big\rangle$$

K is Not Free

Theorem (Stafford (1985))

Let R be a commutative Noetherian domain. Suppose $S = R[x, x^{-1}; \sigma]$ is a skew Laurent extension of R with elements $r, s \in R$ such that

1. *r is not a unit in S,*
2. *$rS + (x + s)S = S$,*
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Then the S-module $K = \{f \in S \mid rf \in (x + s)S\}$ is a non-free, stably free right ideal of S, satisfying $K \oplus S \cong S \oplus S$.

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(Pf)

$r = 1 + y + y^3$ is not a unit since r is not a monomial.

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$\{r, x + s\}$ generates $\mathbb{Z}G$ since π is surjective.

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$$(1 + y + y^3) \nmid (y^6 + 2y^5 + y^4 + 2y^3 + 2y^2 + 1)$$

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Hence, K is a non-free, stably free module over $\mathbb{Z}G$.$\square$

$(G, 2)$-Complexes

Definition

Let G be a group. A $(G, 2)$-**complex** is a 2-dimensional CW-complex with fundamental group G.

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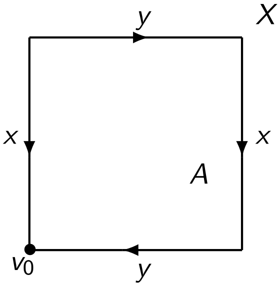
Let X be the CW-complex of the Klein Bottle. Thus, X is a $(G, 2)$-complex with $G = \pi_1(X, v_0)$.



Euler Characteristic

Definition

Let Y be a finite 2-dimensional CW-complex. Then the **Euler Characteristic** of Y, denoted $\chi(Y)$, is the alternating sum $\sum_{k=0}^{2}(-1)^k c_i$ where c_k is the number of k-cells in Y.

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Theorem (Asphericity)

The Klein bottle is the only complex with $\chi(X) = 0$*, which is the minimum Euler characteristic for any* $(G, 2)$*-complex, up to homotopy.*

Algebraic $(G, 2)$-complexes

Definition

An **algebraic** $(G, 2)$-**complex** is an exact sequence

$$\mathcal{C}_* \quad : \quad F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

where the F_i are finitely generated, free $\mathbb{Z}G$-modules. The **Euler Characteristic** of $\mathcal{C}_*$, denoted $\chi(\mathcal{C}_*)$, is the alternating sum $\sum_{k=0}^{2} (-1)^k c_i$ where c_k is the rank of F_k in $\mathcal{C}_*$.

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Then,

$$\mathcal{C}_*(\widetilde{X}) \quad : \quad \mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is an algebraic $(G, 2)$-complex.

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Notice that

$$\pi_2(X_1) \cong H_2(\widetilde{X_1}) = \ker(\partial_2 \oplus 0) = \mathbb{Z}G.$$

The Algebraic Complex $\mathcal{K}_*$

We now will construct an algebraic $(G, 2)$-complex with no obvious geometric interpretation. Let $\mathcal{K}_*$ to be the exact sequence

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Therefore,

$$\mathcal{K}_* \not\simeq \mathcal{C}_*(X_1).$$

Geometric Realization

Theorem

There exist chain-homotopically distinct, algebraic $(G, 2)$*-complexes with Euler characteristic 1, where* G *is the fundamental group of the Klein bottle.*

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Question

Does $\mathcal{K}_*$ arise as an algebraic complex of some geometric $(G, 2)$-complex?