Incorporating Discontinuous Boundaries in an ERT Inversion Through Regularization Operators

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Abstract Electrical resistivity tomography has proved to be a useful tool for many subsurface investigations. However, this nonlinear inverse problem is severely under-determined and consequently smoothness constraints are commonly implemented with least squares to make the problem solvable. This approach is limited by its inability to produce discontinuities in the model parameters. In practice, sharp delineations can be recovered with this inversion approach by applying appropriate weights or covariance matrices which relax the constraint at different regions within the model parameters. However, little work has been devoted to understanding the theory, implications, and characteristics of relaxing the constraint. In this work, we provide a general and thorough interpretation of how to incorporate spatial discontinuities into the constraint through a separate operator that we call the regularization operator. We use this general interpretation to explore the methodology, theory, and characteristics of incorporating prior information into the regularization process. We show how and why this regularization operator can be used to incorporate a boundary into the inversion process, and apply this methodology to a variety of synthetic ERT examples, where we investigate the characteristics of discontinuous inversions and the type of constraint applied. Not surprisingly, our results show that the implementation of a proper boundary estimate into the regularization operator significantly improves the accuracy of the inverted model primarily by removing much of the smearing that typically occurs across the boundary. Not only do we show how to properly implement a boundary...
estimate, we also explain why using the derivative as prior information is more advantageous than using an initial parameter estimate. We show the difference between using 1st and 2nd derivative operators. In particular, the former produces model parameters with constant variability within the subregions, while the latter will yield linear variability. These findings can be combined with existing methods for estimating covariance matrices.

**Keywords** Electrical Resistivity Tomography · Nonlinear Inverse Problems · Discontinuous Inversions · Regularization

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### 1 Introduction

Electrical resistivity tomography has proven to be useful for a wide variety of subsurface investigations including salt water intrusions [20, 8, 9], tracer studies [19, 11, 15, 21], and vadose zone soil moisture analysis [3, 1, 14]. Imaging subsurface structures from ERT measurements requires the solution of an inverse problem. Most inverse methods require regularization and typically smoothing constraints are implemented to produce inverted results [2, 4]. Smoothing constraints are useful for producing general trends interpretable at large spatial scales, and have proven to be especially useful when temporal data is available to compare changes in the subsurface with time.

The implementation of standard smoothness constraints is not ideal for most ERT studies because they prevent the ability to produce model parameters with discontinuities. Several ERT studies have a known discontinuous boundary present in the subsurface such as permafrost, bedrock, or the water table. An accurate, quantitative analysis of parameters such as soil moisture requires inverse methods that allow the parameters to be discontinuous.

There have been some studies that promote and implement the use of discontinuities in ERT inversion. This includes semivariogram constraints [9, 10], joint inversion [5], and covariance matrices [17]. The most direct approach of incorporating discontinuities is by relaxing the regularization constraint [7]. The location of where to relax the constraint typically comes from boundary estimates derived from some other data such as ground penetrating radar (GPR) [6], well data [10], or known man-made structures in the subsurface [18]. However, little has been done to develop the theory of discontinuous inversions, which is required in order to gain further understanding into the nature of these types of inversions.

In this work, we present an encompassing investigation into the methodology, theory, and characteristics of discontinuous inversions. We suggest that discontinuous inversions can be viewed as a special case of a more general regularization matrix that acts within the constraint term, which spatially applies relative degrees of regularization to the inverted model parameters. We combine the discontinuous inversion with an approach similar to Occam’s method
for nonlinear inversion, but apply a statistical test approach for choosing the regularization parameters, rather than using the discrepancy principle.

We begin by introducing the forward and inverse models in Section 1. In Section 2 we explain the effect of typical smoothness constrains in an inversion and how the regularization operator allows for discontinuities. This theory is illustrated on idealized discontinuous parameter scenarios. We describe the implementation of the regularization operator for two dimensional ERT inverse problems in Section 3. In Section 4 we present an a variety of synthetic models that show the characteristics of the different constraints used with the regularization operator and standard smoothness inversions. We elaborate on these results in Section 5.

1.1 The Forward Model: Ohm’s Law

Resistivity surveys are used to approximate two or three dimensional features within the subsurface. This is accomplished by distributing either a line or grid of electrodes along the surface. A pair of transmitting electrodes are used to induce a low frequency, alternating current, into the ground. A separate pair of receiver electrodes measure the potential difference elsewhere. The ratio of the injected current and the measured voltage, combined with the electrode geometry, yields the measured result known as the apparent resistivity and is given by:

\[
\rho_a = \frac{2\pi \Delta V}{i \kappa}
\]

where

\[
\kappa = \frac{1}{(1/\Delta x_{AM} - 1/\Delta x_{AN} - 1/\Delta x_{BM} + 1/\Delta x_{BN})}.
\]

The variable \( \kappa \) is known as the geometric factor that contains all of the geometrical information of the electrodes, \( \Delta V \) is the electrical potential different across the receiver electrode pair, \( i \) is the injection current, and \( \Delta x_{AN} \) represents the distances between the current electrodes A and B and the voltage electrodes M and N, see Figure 1. The apparent resistivity is equivalent to what would be measured for a homogeneous subsurface. These values are treated as observational data and can be inverted to yield an approximate model of the subsurface resistivity \( \rho \).

The governing equation that provides the relationship between the resistivity in the subsurface and the measured apparent resistivity is Ohm’s Law:

\[
\nabla \cdot [\sigma(r) \nabla V(r)] = i[\delta(r - r_A) - \delta(r - r_B)]
\]

where \( \sigma = 1/\rho \) is the conductivity. The variables \( r_A \) and \( r_B \) represent the locations of current source electrodes A and B, respectively.

This study implements a Matlab-based 2.5D forward model developed by Adam Pidlisecky and Rosemary Knight [16], where the Fourier-transform is
used in the y direction so that resistivity varies along a horizontal and vertical dimensions while the y dimension has constrained variability. The two-dimensional model takes the form:

$$\frac{\partial}{\partial x} \left[ \sigma(x, z) \frac{\partial V(x, z)}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \sigma(x, z) \frac{\partial V(x, z)}{\partial z} \right] - \lambda_y^2 \sigma(x, z) V(x, z) = i \left[ \delta(x - x_A) - \delta(x - x_B) \right]$$

where $\lambda_y$ represents the spatial wavenumber in the y direction, which represents the degree of variability in the y direction. The input to the forward model takes a given resistivity model and current source distribution to yield electrical potential at all locations. This code was altered to allow conversions from electrical potential values at receiver electrode positions to apparent resistivity observations from given electrode combinations for wenner-alpha arrays [22].

1.2 The Inversion

The forward model for resistivity requires an input of resistivity model parameters:

$$m = \rho(x, z), \quad m \in \mathbb{R}^n.$$ 

The associated output yields a set of apparent resistivity values:

$$F[m] = \rho_a, \quad F[m] \in \mathbb{R}^m.$$ 

Therefore, the inversion of resistivity data takes measured apparent resistivities, $d^{obs}$, as input so that

$$d^{obs} = F[m] + \epsilon, \quad d^{obs} \in \mathbb{R}^m$$

with noise $\epsilon$. In this study, we use synthetic measurements and represent $\epsilon$ as Gaussian noise. Even though this assumption may not be true for particular scenarios encountered in the field, it is a common assumption and will be applied in this study.

The output is an estimate of the true resistivity, $m$. This inversion problem is ill-posed because model parameters are overdetermined near the observational data, underdetermined far away from the observational data, and it is typical for significant noise to be present in the observations. We address ill-posedness by incorporating additional criteria. This is often viewed as regularization in the form of a smoothness constraint and commonly referred to as smooth-model inversions.

A smooth-model inversion attempts to minimize an objective function $\phi$ containing two terms: the data residual $\phi_d$ and regularization residual $\phi_r$:

$$\phi(m) = \phi_d(m) + \alpha^2 \phi_r(m) = \left| | W_d(d^{obs} - F[m]) | \right|^2 + \alpha^2 || L_p (m - m_{ref}) ||^2$$

where $m_{ref}$ is the given reference model, $\alpha$ the regularization parameter, $W_d$ the data weighting matrix, and $L_p$ represents the $p_{th}$ derivative operator.
2 Discontinuous Inversions

2.1 The Constraint

Smooth model inversion can be viewed as constrained optimization, i.e.

\[ \min \phi_d(m), \quad s.t. \quad \phi_m \leq \delta. \]

The goal of the constraint term in regularization is to stabilize the inversion problem such that small perturbations in the data yields small perturbations in the inverted model. Yet the constraint term can also be viewed as adding prior information, where the type of constraint should reflect something already known about the true model.

Regularization methods have found much success by enforcing a smoothness constraint, most commonly by requiring that the \( p \)-th derivative of the model parameters is minimized. In this work we will consider the three of the most common types of smoothness constraints: the 0-th, 1-st, and 2-nd derivatives. By viewing the regularization terms as constraints containing prior information, we will develop a more detailed understanding for producing discontinuous parameters with \( p \)-th derivative constraints.

When the zeroth derivative is used, the derivative operator becomes the identity matrix:

\[ L_0 = I. \]

With this approach, the reference model \( m_{ref} \) is used as the primary constraint, i.e.:

\[ ||L_0(m - m_{ref})||^2 \leq \delta \]

is prior information on the model parameters. If the reference model is a good estimate, then the inversion yields acceptable results. However, a good reference model requires both the spatial information of any boundaries within the subsurface, as well as estimates of the resistivity values on either side of a given boundary. This is a large amount of information required, which is why this method is seldom used in most modern ERT processing. The other difficulty with this approach is that if the reference model is incorrect, bias is introduced in the inversion without correct weights. Knowledge of the reference model’s accuracy is again a substantial amount of prior information that typically is not available. Finally, if one seeks to provide a detailed reference model, it takes away the purpose of inversion in the first place, since the inversion will merely prefer models similar to the reference model and have less influence on the data. We find it is not practical to incorporate the presence of a discontinuity into this type of constraint. An appropriate weight will result in a parameter chosen purely by the data misfit term, and not reflect the constraint on the discontinuity.

A second approach is to constrain the 1-st derivative through \( L_1 \). The reference model for this type of constraint is typically a zero vector or some constant value. This type of constraint gives the opportunity to specify a discontinuity in the model, rather than the actual value of the model. It is beneficial in
that it requires less prior information and yield good stability for the inversion problem. To illustrate this point we provide Figure 2, which shows an example one dimensional reference model along with its 1\textsuperscript{st} and 2\textsuperscript{nd} derivatives. From this Figure shows how the 0\textsuperscript{th} derivative requires more information relative to the 1\textsuperscript{st} and 2\textsuperscript{nd} derivatives which are mostly zero everywhere. Minimizing the 1\textsuperscript{st} derivative of the model parameters should lead the inversion algorithm toward inverted models with no variability (constant values). If a discontinuity is incorporated into this type of constraint, then the only prior or additional information that is added is merely the spatial distribution of boundaries, and does not require an estimate of the magnitude of the change in values across a given boundary. This will also allow for the inversion process to yield a discontinuity separating two sub-regions, each without any variability. Therefore, if the subsurface consists of discontinuities delineating regions with very little variability, then using the 1\textsuperscript{st} derivative as a constraint and allowing discontinuities within it should produce significantly more realistic results only by adding minimal information, that is, the spatial location of where the boundary occurs.

The second derivative, $L_2$, is another type of constraint that has found much success in geophysical applications including ERT processing. Minimizing the 2\textsuperscript{nd} derivative should lead the inversion algorithm to produce inverted model parameters that also gives the opportunity to specify discontinuities and allows the model parameters to vary linearly. With the incorporation of discontinuities, the results tend to have sharp boundaries separating between sub-regions which vary linearly. Therefore, if it is known that sharp boundaries exist within the true model separating regions which vary linearly, then the 2\textsuperscript{nd} derivative would likely be the optimal constraint to use in the inversion process.

2.2 The Regularization Operator

Appropriate weights or covariance matrices allow the ability to relax the constraint at different regions within the model parameters. We suggest a more general interpretation to this approach, which is to contain all of the spatial variability which controls the degree to which the constraint is applied within the model parameters into a separate operator. We call this matrix the regularization operator and notate it as $R$. Depending on the type of constraint used, the regularization matrix operates on either the 1\textsuperscript{st} or 2\textsuperscript{nd} derivative operators. This operator $R$ is incorporated as follows

$$\phi(m) = ||W_d(d - F[m])||_2^2 + \alpha^2||RL_p m||_2^2$$
Discontinuous Boundaries and the R Operator

where

\[
L_1 = \frac{1}{\Delta x} \begin{pmatrix}
-1 & 1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & -1 & 1 & 0 \\
0 & \ldots & \ldots & \ldots & 0
\end{pmatrix}
\]

\[
L_2 = \frac{1}{\Delta x^2} \begin{pmatrix}
-1 & 2 & 1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 1 & -2 & 1 & 0 \\
0 & \ldots & \ldots & 0 & 0 & \ddots \\
0 & \ldots & \ldots & \ldots & 0 & 0
\end{pmatrix}
\]

and

\[
R = \text{diag}(r_1, r_2, \ldots, r_n), \quad r_i = 0 \text{ or } 1.
\]

If \( R = I \) the result is the typical smoothness constraint where one encounters either a 1st or 2nd derivative constraint. Alternatively, if there is a sharp discontinuity, \( r_i = 0 \) is chosen at that location indicating that there is no smoothness at this particular spatial location. A value of one is assigned to all other values of \( r_i \). Therefore the regularization operator provides information regarding where smoothness is applied spatially to the model parameters and only data inform the value of the parameters at the discontinuity.

For example, in the one-dimensional case where the model parameters consist of six components given by:

\[
m = (m_1, m_2, m_3, m_4, m_5, m_6)^T
\]

applying the first derivative operator \( L_1 \) results in the constraint

\[
L_1 m = \frac{1}{\Delta x} \begin{pmatrix}
m_2 - m_1 \\
m_3 - m_2 \\
m_4 - m_3 \\
m_5 - m_4 \\
m_6 - m_5 \\
0
\end{pmatrix}
\]

This gives smooth parameters across the domain.

Suppose a sharp transition occurs in-between \( m_2 \) and \( m_3 \), and \( m_4 \) and \( m_5 \):

\[
R = \text{diag}(1, 0, 1, 0, 1, 1)
\]

so that the total product yields:

\[
RL_1 m = \frac{1}{\Delta x} \begin{pmatrix}
m_2 - m_1 \\
0 \\
m_4 - m_3 \\
0 \\
m_6 - m_5 \\
0
\end{pmatrix}
\]

In this case the 1st derivative constraint is applied everywhere except along the transition between \( m_2 \) and \( m_3 \) as well as a second discontinuity between \( m_4 \) and \( m_5 \).
and \( m_5 \). If the regularization matrix is included as part of the constraint term within the inversion process, it will result in an array of inverted parameters with a discontinuity delineating between two smooth regions on either side.

By comparing the results between \( L_1 m \) and \( RL_1 m \), it is evident that by incorporating prior information into the regularization matrix, the degrees of freedom in the regularization residual is decreased. In fact, the number of discontinuous points included within the regularization matrix, is the number of degrees of freedom that is lost.

### 2.3 Toy Example

Here we study the effects of 1\(^{st}\) and 2\(^{nd}\) derivative constraints with or without the incorporation of the regularization operator \( R \) on a canonical three layered subsurface model. It is assumed that there is a large weight on the constraint term so that we can understand the constraint’s effect on the inversion. We denote the inverted parameters by:

\[
\begin{align*}
m_{RL} &= \text{argmin}(\phi(m)) & \text{for } R \neq I \\
m_{L} &= \text{argmin}(\phi(m)) & \text{for } R = I
\end{align*}
\]

Figure 3(a) gives an idealized example of a true model with constant variability in the subregions, while Figure 3(b) reflects linear variability in the subregions.

The rows in each of Figures 4-7 show the idealized inverted parameters in the 1\(^{st}\) column, along with their 1\(^{st}\) and 2\(^{nd}\) derivatives in the 2\(^{nd}\) and 3\(^{rd}\) columns, respectively. Figures 4 and 6 give idealized inverted results with the standard smoothness constraints \( L_1 m \) and \( L_2 m \) while the effect of the regularization operator through discontinuous smoothness constraints \( RL_1 m \) and \( RL_2 m \) is shown in Figures 5 and 7. When \( R \neq I \) is applied as the constraint, \( m_{RL2} \) only fits the data at the indicated discontinuities and this is indicated by the open circles in Figures 5 and 7.

The first columns in Figures 4 and 6 show how inverted results with 1\(^{st}\) and 2\(^{nd}\) derivative constraints merely provide a constant or linear average of parameter estimates, respectively. Alternatively, the first columns in Figures 5 and 7 show how the regularization operator applied to both 1\(^{st}\) and 2\(^{nd}\) derivative constraints can yield inverted parameters that both include the presence of a boundary as well as reflect constant or linear variability within each subregion.

The idealized results in Figures 4-7 are justified by considering the last two columns in each of the Figures. The 2\(^{nd}\) derivative constraint \( ||L_2 m|| \), reflected in row 2, enforces \( m_{L2} \approx 0 \) since the reference model is inherently \( L_2 m_{ref} = 0 \). Hence values of zero are illustrated in (f) for all of the Figures. Figure (e) is inferred from (f) by integration, and hence the first derivative is simply taken as a constant since the second derivative is zero. The 1\(^{st}\) derivative constraint \( ||L_1 m|| \) is reflected in row 1. It enforces \( m_{L1} \approx 0 \) hence zero values are in part (b) of the Figures. Part (c) of each Figure is also zero since it is inferred by differentiation of part (b).
The toy examples with the standard smoothness constraint in Figures 4 and 6 show how the inverted results cannot detect constant vs linear variation in the subregions. However, the 2\textsuperscript{nd} derivative constraint can detect linear variation across the domain while the 1\textsuperscript{st} derivative always gives constant parameters. Both 1\textsuperscript{st} and 2\textsuperscript{nd} derivative constraints are improved with \( R \neq I \) as indicated in Figures 5 and 7. In Figure 5 the RL\textsubscript{2} constraint will produce constant values for the inverted model parameters. The value of the constants will be determined by the data, and hence in this toy example it is assumed that the inverted result will have a strong sensitivity to the data. We will give inverted results in Section 4 that illustrates this point. Note that we have also assumed a strong sensitivity to the data in Figures 4 and 6 in inferring the antiderivatives \( m'_{l_2}, m_{l_1}, \) and \( m_{l_2}. \) Even though these are idealized results, they show what is possible or not possible with \( L_1, L_2, R L_1, \) and \( R L_2 \) constraints. In particular it shows how the 2\textsuperscript{nd} derivative constraint is the best type of constraint if there is expected to be linear variation.

It is important to emphasize that the choice of using the 1\textsuperscript{st} or 2\textsuperscript{nd} derivative as the constraint term rests on two factors: 1) the amount (or lack) of variability within each region separated by the discontinuities, and 2) the degree of sensitivity that the observational data have on each of these regions. From our toy example, we see that a constraint utilizing the 1\textsuperscript{st} derivative would perform optimal if no variability exists within each subregion. This is because a minimization of the 1\textsuperscript{st} derivative would yield constant values for the model parameters. Conversely, a constraint utilizing the 2\textsuperscript{nd} derivative would perform optimally if each of the regions vary linearly in some direction. This is because of the effects of minimizing the 2\textsuperscript{nd} derivative results in constant values in the 1\textsuperscript{st} derivative, and linear variability for the model parameters.

Due to the dependence of the sensitivity of the data to the model parameters, no theoretical absolute dictates which approach would perform consistently better when applied to a problem where the variability between the model parameters do not vary. Clearly the 1\textsuperscript{st} derivative would be a good choice, however, the 2\textsuperscript{nd} derivative could also yield the same results. The reasoning behind this is because minimizing the 2\textsuperscript{nd} derivative could also result in a constant value of zero in the 1\textsuperscript{st} derivative, which would result in regions of constant values in each region in the model parameters. This performance would be purely dependent on the spatial sensitivity between the observations and each region in the model parameters. With greater sensitivity, the 2\textsuperscript{nd} derivative constraint would perform just as good as the 1\textsuperscript{st} derivative constraint for model parameters with discontinuities separating regions that have no variability. In this case, the 2\textsuperscript{nd} derivative potentially provides the optimal results in either constant or linear variability within each of the regions. As long as the data was acquired so that the observations are sensitive to each of these regions, the 2\textsuperscript{nd} derivative should provide the best results. Therefore, though this toy example we have not only explained the effect of the regularization operator \( R, \) but have also justified why most geophysical applications find success with the 2\textsuperscript{nd} derivative constraint. We will verify these conclusions in an ERT synthetic inversion in Section 4.
3 Implementation of the Regularization Operator

3.1 Assumptions

It is important to outline the underlying assumptions that this methodology utilizes. The first and foremost assumption made is that a discontinuity is somewhere present within the model parameters. In the case of geophysical investigations, this would imply that the geophysicists knows that there is a discontinuous boundary delineating between regions within the subsurface with vastly different geophysical properties. The second assumption is that the character of the discontinuity is sharp such that the distance of the transition that occurs between each region is small relative to the spatial resolution of the model parameters. A good example occurs frequently with electrical resistivity tomography, wherein the spatial resolution of the model parameters are on the order of the electrode separation. This distance is typically much greater than the vertical transition across the water table, which is on the order of centimeters, so that the transition of electrical properties across this boundary is effectively discontinuous relative to the discretization of the inverted resistivity profiles. The third assumption that is required is that the observational data must have sensitivity to each region that the discontinuity delineates. If the data is not sensitive to any regions separated by a boundary, then the data is not likely sensitive to the boundary itself, and implementing a regularization operator into the inversion process will not provide a significant benefit to the problem.

3.2 The Two-Dimensional Problem

Our synthetic results include inversions of electrical resistivity tomography for a variety of simple resistivity distributions within the subsurface. The models in these problems are two dimensional and the inversion problem applies Occam’s method, which takes the form:

$$
\phi(m) = ||W_d(d - F[m])||_2^2 + \alpha^2[||L_{px}m||_2^2 + ||L_{pz}m||_2^2].
$$

For this problem, the regularization matrix becomes two matrices that determine the relative amount of the horizontal and vertical smoothing that occurs spatially:

$$
\phi(m) = ||W_d(d - F[m])||_2^2 + \alpha^2[||R_xL_{px}m||_2^2 + ||R_zL_{pz}m||_2^2].
$$

3.3 Occam’s Method

Occam’s method is a commonly used technique in geophysics for nonlinear inversions [2], [4]. Using an initial estimate $m(0)$, Taylor’s Theorem is used to
give a local linear approximation of $F[m]$ which we combine with the regularization operators so that the objective function becomes

$$
\phi_k(m_{(k+1)}) = \phi_d(k) + \alpha^2[\phi_{mx}(k) + \phi_{mz}(k)] = \|W_d(d(k) - J_k(m_{(k+1)})\|_2^2 + \alpha^2[\|L_pz(m_{(k+1)} - m_{ref})\|_2^2 + \|R_xL_{px}(m_{(k+1)} - m_{ref})\|_2^2]
$$

with

$$d_k = d^{obs}(k) - F[m_{(k)}] + J_k(m_{(k)}).$$

At each iterate, the minimum of the linear cost function occurs at:

$$m_{(k+1)} = \left[\alpha^2(L_p^T R_p^T R_x L_p + L_p^T R_x^T R_x L_p) + J_k^T W_d^{-1} W_d d(k) \right]^{-1} [J_k^T W_d^{-1} W_d d(k) - \alpha^2(L_p^T R_p^T R_x L_p + L_p^T R_x^T R_x L_p)m_{ref}].$$

Occam’s method uses the discrepancy principle in order to find the regularization parameter $\alpha$, i.e. $\alpha$ is found so that:

$$\phi_d(m_{(k+1)}) \leq \Delta.$$

The choice of $\Delta$ is often left as an open question. If the data misfit is considered as a $\chi^2$ random variable, its mean can be used for $\Delta$. In this case the degrees of freedom, and its mean, is $m - n$. For $m \approx n$ this approach is not practical and the $\chi^2$ method [13, 12] that uses the regularized residual can be used. In the $\chi^2$ method $\alpha$ is found so that:

$$\phi(m_{(k+1)}) \approx \chi^2_m = m.$$

Not only does the $\chi^2$ method give a statistical justification for the choice of $\alpha$, we suggest that it is important to include this sensitivity towards the constraint when choosing the regularization parameter because it will be more sensitive to the type of constraint used within the inversion process. If a good estimate of the boundary location is properly implemented into the regularization matrix, then $\|RL_p m_{true}\|_2^2$ will be small relative to other potential estimates of the boundary contained in $R$. This implies inverted results should reflect a greater weight on minimizing the regularization residual. Therefore, if the presence of a sharp boundary is known prior to data acquisition or processing but not the exact location, then a variety of boundary estimates will provide different regularization parameters, which will yield different inverted results. The inverted result with the largest regularization parameter found by the $\chi^2$ test will indicate that the associated boundary estimate is the most probable. This will be explored in future work.
4 Numerical Results

In this section, we provide inverted results from three fundamental model types: 1) two-layered models in Figure 8, 2) anomaly models in Figure 9, and 3) sinusoidal models in Figure 10. By considering both constant and linear variability in all three model types, a total of seven models were used for our synthetic analysis. These model types represent the challenge in ERT inversion with respect to the sensitivity of the observational data to 1) subregions distant from the observations, 2) subregions with small anomalies, and 3) complex subregion boundary geometry. We compare results from standard 1st and 2nd derivative constraint inversions to those that use discontinuous regularization operators. In each model 0.1% Gaussian noise was added to each data sample to create the synthetic observations. Four different resistivity values were estimated from each of the seven sets of synthetic data, and are denoted by $m_{L1}$, $m_{L2}$, $m_{RL1}$, and $m_{RL2}$ as defined in Section 2.3.

4.1 Two-Layered Models

The two-layered models in Figure 8 contain less sensitive subregions distant from the observations recorded at the surface. The inverted results are provided in Figures 11 - 13. Figure 11 is the simplest of the three models and we see that 1st and 2nd derivative constraints exhibit significant smearing in Figures 11 (a)-(b). With knowledge of the boundary location at 7m depth, both the 1st and 2nd derivative constraints provide a more accurate representation of the true model in Figures 11 (c)-(d). The ability of the regularization operator to delineate different subregions can be observed more clearly by producing vertical slices that represent the average horizontal resistivity with depth provided in Figures 14 (a)-(b). These Figures show the smearing located near the discontinuity that takes place with standard inversions, whereas the regularization operator show nearly identical results with the true model.

Two different amounts of linear variability were added to the two layered model which we call moderate and strong linear variability. Values with moderate linear variability range between 2000 $\Omega m$ - 4000 $\Omega m$ in the upper subregion and values between 400 $\Omega m$ - 1000 $\Omega m$ in the lower region and are given in Figure 8 (b). For the strong linear variability model, values in the upper subregion are the same as the moderate linear model, while values in the lower region range between 400 $\Omega m$ - 6000 $\Omega m$ which is given in Figure 8 (c). The inverted results with the moderate variability model in Figure 12 are similar to that with constant variation in that smearing is significantly reduced when knowledge of the boundary is incorporated through regularization operators. A closer inspection of the vertical slices in Figures 14 (c)-(d) however show that the exact behavior of the parameters is not captured with the regularization operator. The 2nd derivative constraint in 14 (d) with the regularization operator is slightly better than the 1st derivative operator in 14 (c). This is subtle improvement in performance is located in the upper region near
the boundary, where $m_{RL_2}$ shows slightly more linear variability than $m_{RL_2}$. In either case, the region where the regularization operator yields more optimal results occurs at each side of the boundary, where standard smoothness inversions biases this region by smearing the two subregions together.

Inverted results in Figure 13 with strong variability have stronger smearing than with moderate linearity. The smearing in the standard inversions is so severe that it is difficult to delineate the location of the discontinuity as well as detect the presence of linear variability. With the implementation of the proper discontinuity through the regularization operator, linearity is clearly observed at least within the upper subregion. However, closer inspection in Figures 14 (e)-(f) show a lack of linear variability in the lower subregion for all inverted types relative to the true model. This lack of convergence is partially present in the upper subregion, however it is most noticeable in the lower subregion. This lack of convergence is likely due to a lack of sensitivity in the data, which implies that the degree of variability could limit the performance of the regularization operator. In these inverted results we find that $m_{RL_2}$ performed better than $m_{RL_1}$. This difference in performance is evident in both subregions, where $m_{RL_2}$ produced more linearity similar to the true model.

4.2 Anomaly Models

The anomaly models in Figure 9 contain a less sensitive subregion that covers a small area relative to the rest of the model. The inverted results are provided in Figures 15 and 16. The results in Figure 15 (a)-(b) show the continued pattern of smearing produced from standard inversions, whereas the incorporation of the proper boundary significantly removes this effect to yield near exact results in Figures 15 (c)-(d). The vertical slices in Figures 17 (a)-(b) show a drastic improvement in performance with the implementation of the regularization operator when compared to the standard inverted results. These Figures also show that the $RL_1$ constraint had the best convergence to the true model than the $RL_2$ constraint, which produced artificial linear variability in the region below the anomaly. More information about the accuracy of the inverted parameters can be obtained by also viewing horizontal slices that represent the average vertical resistivities, provided in Figure 18. For the inverted results with constant variability in Figures 18 (a)-(b), we find near exact convergence of both $RL_1$ and $RL_2$ constraints when compared to the true model.

The inverted results from the anomaly model with moderate linear variability is provided in Figure 16. The standard inverted parameters show the same smearing pattern, although both 1$st$ and 2$nd$ derivative constraints are able to produce some degree of linearity reflected in the true model. However, when the boundary location is provided in the inversion, smearing is greatly reduced and the degree of linearity has a stronger presence. The sensitivity to linearity is more clearly seen in the vertical slices provided in Figures 17 (c)-(d). These results show that the standard inversions indeed recover a significant amount of the linear variability, however the degree of accuracy is not
The results also show that $m_{RL_1}$ performed slightly better than $m_{RL_2}$ particularly in the region just below the anomaly. The horizontal results in Figures 18 (c)-(d) show how significant the degree of improvement is obtained when the discontinuity is incorporated into the inversion.

### 4.3 Sinusoidal Models

The primary challenge with the sinusoidal models is the presence of a complex boundary geometry. The inverted results are provided in Figures 19 and 20. The standard inversions from Figures 19 show the continued presence of smeared results and can only provide a general interpretation of the true model. However, the inverted results that contain the proper discontinuity removed a large degree of smearing to yield near exact results to the true model. This is true for both 1$^{st}$ and 2$^{nd}$ derivative results, where the degree to which they converged to the true model is comparable. When moderate linear variability is introduced, the results in Figure 20 show even more benefit for incorporating discontinuities into the inversion process. Not only is there a significant amount of smearing present in the standard inversions, but there is no clear indication of linear variability. However, the discontinuous inversions are able to recover most of the linear variability, particularly in the upper subregion. From these results, the $RL_2$ constraint performed better than the $RL_1$ constraint by producing more linear variability in accordance with the true model.

### 5 Discussion

These results show that the implementation of discontinuities in the inversion process is likely to yield more optimal results, regardless of partial sensitivity between the observational data and subregions within the model parameters due to either subregions being far away from the observations, subregions with a small area of extent, or even a complex boundary geometry. In fact, the degree of sensitivity appears to remove the amount gained by incorporating additional information in the form of a discontinuity. The results also show how these sensitivity issues limit the performance of implementing a discontinuity into a given inversion. We observe this in the two-layered model with a high degree of linear variability, where a bias is present in the lower subregion even with the implementation of the $RL_2$ constraint term (Figure 14 (f)).

We also found a small degree of variability between the performance of the 1$^{st}$ and 2$^{nd}$ derivative constraints. In most cases, these constraints produced very similar results, where in some cases, the 1$^{st}$ derivative constrained inversions performed better where the true models had little variability within each subregion, while in other cases, the 2$^{nd}$ derivative constrained inversions performed better where the true models had significant linear variability. This
is consistent with our toy example, where the 1\textsuperscript{st} derivative constraint produced constant variability and the 2\textsuperscript{nd} derivative constraint produced linear variability. The primary difference between the toy example and the synthetic results was the degree of sensitivity, where the sensitivity was ideal in the toy example, and lesser in the synthetic results. In fact, we suggest that this partial sensitivity explains why certain models, such as the two-layered model and constant model with constant variability (Figures 14 (a)-(b) and 18(a)-(b)), had the 2\textsuperscript{nd} derivative constraint produce linear variability even though the true model had only constant variability within each subregion.

6 Conclusions and Future Work

In this study, we present a general form of least squares inversion appropriate for implementing discontinuities as prior knowledge. This is performed by incorporating a matrix \textbf{R} within the constraint term of the inversion that contains zeros and ones. The regularization operator \textbf{R} is diagonal with ones everywhere except at locations representing a discontinuity. Setting the \textit{i}th value along the diagonal of \textbf{R} to zero will allow the presence of a discontinuity between \(m_i\) and \(m_{i+1}\). Multiplying the regularization term by \textbf{R} removes the influence of the constraint on targeted regions for one dimensional problems and we extended the approach to two dimensions through synthetic examples. When the location of the boundary is known, this approach gives significantly improved results.

An understanding of the influence of \textbf{R} on the regularization term gives insight to the effect of regularizing with a first derivative (\(L_1\)) or second derivative (\(L_2\)) constraint. We firstly noted that \(L_1\) and \(L_2\) regularization are more practical than \(L_0\) for discontinuous inversions since they require less prior information. Secondly we found that the \(L_1\) constraint is the best choice when the true model contains constant variability in subregions. Alternatively, if there is linear variability in subregions the \(L_2\) constraint is optimal. In particular, we showed that if there is a lack of sensitivity between the observations and particular subregions, then \(L_1\) will necessarily produce constant variability while \(L_2\) will produce linear variability in the subregions.

An in depth analysis of the incorporation of sharp discontinuities with least squares is necessary before applying it to more practical problems. Least squares necessarily produces continuous, smooth results. However, by incorporating a matrix \textbf{R}, piecewise, continuous results are obtained and hence we have shown exactly how least squares can capture discontinuities. When dealing with real data where prior knowledge is not perfectly known (i.e. the location of a boundary), more elaborate techniques are required to appropriately weight such prior information. This is typically accomplished by a covariance matrix that acts on the constraint in the regularization term. For example, it has been shown that the covariance matrix can be formed from variogram estimates of the model parameters [9]. However, this approach finds a covariance matrix for the 0\textsuperscript{th} derivative of the model parameters. We have
shown it is more advantageous to use 1\textsuperscript{st} and 2\textsuperscript{nd} derivative constraints for problems with discontinuities. Therefore, a more appropriate implementation in this setting would be to construct a variogram of the 1\textsuperscript{st} or 2\textsuperscript{nd} derivative of the model parameters. This will be the focus of future work, where we incorporate inexact boundaries with real data.

References

7. Günther, T., and Rücker, C. A general approach for introducing information into inversion and examples from dc resistivity inversion. 10th Annual European Meeting of Environmental and Engineering Geophysics, EAGE. Extended Abstract.

Fig. 1: The Wenner-alpha electrode configuration.

Fig. 2: An example reference model and its 1st and 2nd derivatives.

Fig. 3: True parameters for the layered toy example.
Fig. 4: Inverted parameters and their derivatives in constant variability toy example, Figure 3(a). $L_1$ enforced exactly in (a)-(c), $L_2$ constraint enforced exactly in (d)-(f). Red boxes indicate exact constraint used for inversion.

Fig. 5: Inverted parameters and their derivatives in constant variability toy example, Figure 3(a). $RL_1$ enforced exactly in (a)-(c), $RL_2$ constraint enforced exactly in (d)-(f). Red boxes indicate exact constraint used for inversion.
Fig. 6: Inverted parameters and their derivatives in linear variability toy example, Figure 3(a). $L_1$ enforced exactly in (a)-(c), $L_2$ constraint enforced exactly in (d)-(f). Red boxes indicate exact constraint used for inversion.

Fig. 7: Inverted parameters and their derivatives in linear variability toy example, Figure 3(a). $RL_1$ enforced exactly in (a)-(c), $RL_2$ constraint enforced exactly in (d)-(f). Red boxes indicate exact constraint used for inversion.
(a) Constant Variability

(b) Moderate Linear Variability

(c) Strong Linear Variability

Fig. 8: True resistivity values for a layered subsurface

(a) Constant Variability

(b) Moderate Linear Variability

Fig. 9: True resistivity values for an anomaly model
Discontinuous Boundaries and the R Operator

(a) Constant Variability

(b) Moderate Linear Variability

Fig. 10: True resistivity values for sinusoidal model
Fig. 11: Inverted resistivity values for a layered subsurface with constant variability
Discontinuous Boundaries and the R Operator

Fig. 12: Inverted resistivity values for a layered subsurface with moderate linear variability
Fig. 13: Inverted resistivity values for a layered subsurface with strong linear variability.
(a) Layered model with constant variability: $m_L^1$ and $m_{RL}^1$

(b) Layered model with constant variability: $m_L^2$ and $m_{RL}^2$

(c) Layered model with moderate linear variability: $m_L^1$ and $m_{RL}^1$

(d) Layered model with moderate linear variability: $m_L^2$ and $m_{RL}^2$

(e) Layered model with strong linear variability: $m_L^1$ and $m_{RL}^1$

(f) Layered model with strong linear variability: $m_L^2$ and $m_{RL}^2$

Fig. 14: Average vertical resistivity of inverted models
Fig. 15: Inverted resistivity values for an anomaly model with constant variability
Fig. 16: Inverted resistivity values for anomaly model with moderate linear variability
Anomaly model with constant variability: $m_{L_1}$ and $m_{RL_1}$

Anomaly model with constant variability: $m_{L_2}$ and $m_{RL_2}$

Anomaly model with linear variability: $m_{L_1}$ and $m_{RL_1}$

Anomaly model with linear variability: $m_{L_2}$ and $m_{RL_2}$

Fig. 17: Average vertical resistivity of inverted models
(a) Anomaly model with constant variability: $m_{L_1}$ and $m_{RL_1}$

(b) Anomaly model with constant variability: $m_{L_2}$ and $m_{RL_2}$

(c) Anomaly model with linear variability: $m_{L_1}$ and $m_{RL_1}$

(d) Anomaly model with linear variability: $m_{L_2}$ and $m_{RL_2}$

Fig. 18: Average horizontal resistivity of inverted models
Fig. 19: Inverted resistivity values for sinusoidal model with constant variability
Fig. 20: Inverted resistivity values for sinusoidal model with linear variability