Math 566 Spring '09
Exam 2 Solutions

1.a) $\dfrac{y_{i+1}-2y_i+y_{i-1}}{h^2}+ay_i+by_i^4=0 \qquad y_0=0 \quad y_m=1$

Solve $F(\underline{y})=0$ where $\underline{y}=[y_1\ y_2\ \cdots\ y_{m-1}]^T$ and

$$F=\begin{bmatrix} y_2-2y_1+ah^2y_1+bh^2y_1^4 \\ y_3-2y_2+y_1+ah^2y_2+bh^2y_2^4 \\ \vdots \\ y_{m-1}-2y_{m-2}+y_{m-3}+ah^2y_{m-2}+bh^2y_{m-2}^4 \\ 1-2y_{m-1}+y_{m-2}+ah^2y_{m-1}+bh^2y_{m-1}^4 \end{bmatrix}$$

b) Newtons Method for $F(\underline{y})=0$

$$\underline{y}^{(j+1)}=\underline{y}^{(j)}-J^{-1}(\underline{y}^{(j)})F(\underline{y}^{(j)})$$

At each iteration solve

$$J(\underline{y}^{(j)})\underline{v}=F(\underline{y}^{(j)})$$

then

$$\underline{y}^{(j+1)}=\underline{y}^{(j)}-\underline{v}$$

where

$$J(\underline{y})=\begin{bmatrix} -2+ah^2+4bh^2y_1^3 & 1 & & 0 \\ 1 & -2+ah^2+4bh^2y_2^3 & 1 & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2+ah^2+4bh^2y_{m-1}^3 \end{bmatrix}$$

c) B.C.: $\dfrac{-3y_0+4y_1-y_2}{2h}=0 \Rightarrow y_0=\dfrac{4y_1-y_2}{3}$

and only the 1st element in F, found in a), changes to

$$y_2-2y_1+\frac{4y_1-y_2}{3}+ah^2y_1+bh^2y_1^4$$

2.

$$\frac{u_{i,j+1}-u_{i,j-1}}{2k}+a\,\frac{u_{i+1,j}-u_{i-1,j}}{2h}=f(x_i)$$

$$u_{0,j}=g(t_j) \qquad u_{i,0}=0$$

$\Rightarrow$

$$u_{i,j+1}=u_{i,j-1}-\frac{ak}{h}(u_{i+1,j}-u_{i-1,j})+2kf(x_i)$$

For $\underline{u}^{(j)}=[u_{1j}\ u_{2j}\ \cdots\ u_{m-1,j}]^T$, $\lambda=\frac{ak}{h}$

$$\underline{u}^{(j+1)}=\underline{u}^{(j-1)}-\lambda\begin{bmatrix}0&1&&\\-1&0&-1&\\&&&-1\\&&1&0\end{bmatrix}\underline{u}^{(j)}+k\underline{f}^{(j)}+\begin{bmatrix}-g_j\\0\\\vdots\\0\end{bmatrix}$$

Matrix has e-values: $\mu_j=2\lambda\sqrt{(1)(-1)}\cos\frac{j\pi}{m}=2\lambda i\cos\frac{j\pi}{m}$

$$|\mu_j|=2\lambda\left|\cos\frac{j\pi}{m}\right|<1 \qquad \lambda>0\Rightarrow\lambda<\tfrac{1}{2}$$

or $k<\frac{h}{2a}$

3. a) weak formulation

$$\int_0^2 ((au')' - f)v\,dx = 0$$

where v are basis functions. In symmetric form

$$\int_0^2 ((au')' - f)v\,dx = au'v\Big]_0^2 - \int_0^2 (au'v' - fv)\,dx$$

If basis functions are chosen so that

$$v(2) = v(0) = 0$$

then the weak formulation in symmetric form is

$$\int_0^2 (au'v' - fv)\,dx = 0$$

b) Choose the mesh: $0 = x_0 < x_1 < \ldots < x_m = 2$
and basis functions

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & x_{i-1} \le x \le x_i \\ \dfrac{x_i - x}{h} & x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Then as shown in class the finite Element approximation is a tridiagonal Matrix.