MATH 566 Spring 2009
Exam 1 March 11, 2009

Please begin each problem on a new page, and write your name on each page.

1. (25 pts.) Consider the linear system

$$\begin{aligned} 3x + y + z &= 1 \\ 2x + 5y + 2z &= 3 \\ 2y + 3z &= -2 \end{aligned}$$

   Start with an initial guess of $[0, 0, 0]^T$ and compute

   (a) the second Jacobi iterate.

   (b) the second Gauss-Seidel iterate.

2. (25 pts.) In solving the linear system $\mathbf{Ax} = \mathbf{b}$ iteratively, we can write $\mathbf{A} = \mathbf{P} - \mathbf{N}$ and generate a sequence $\mathbf{x}^{(k)}$ by the formula

$$\mathbf{x}^{(k+1)} = \mathbf{P}^{-1}\mathbf{N}\mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{b}.$$

   Denote the residual $\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A}\mathbf{x}^{(k)}$ and show that the iteration formula can be equivalently expressed as

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{P}^{-1}\mathbf{r}^{(k)}.$$

3. (25 pts.) Use the 4th order Runge-Kutta method to numerically approximate the solution of

$$y' = 2ty + 1, \qquad y(0) = 2$$

   with $h = 0.05$ for two steps.

4. (25 pts.) Consider the linear multistep method for the ODE $y' = f(t, y)$, $y(t_0) = y_0$:

$$y_{n+1} = y_n + \frac{h}{12}\left(23f(t_n, y_n) - 16f(t_{n-1}, y_{n-1}) + 5f(t_{n-2}, y_{n-2})\right).$$

   (a) Find a formula for the local truncation error.

   (b) Is the method consistent? Stable? Will it converge? Prove your answer in each case.

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Solutions

only graded 1st iterate

1. $x = \frac{1}{3}(1-y-z)$
$y = \frac{1}{5}(3-2x-2z)$
$z = \frac{1}{3}(-2-2y)$

(a) $x^{(1)} = \frac{1}{3}$ $\quad y^{(1)} = \frac{3}{5}$ $\quad z^{(1)} = -\frac{2}{3}$

$x^{(2)} = \frac{1}{3}(1 - \frac{3}{5} + \frac{2}{3}) = \frac{16}{45}$

$y^{(2)} = \frac{1}{5}(3 - \frac{2}{3} + \frac{4}{3}) = \frac{11}{15}$

$z^{(2)} = \frac{1}{3}(-2 - \frac{6}{5}) = -\frac{16}{15}$

(b) $x^{(1)} = \frac{1}{3}$

$y^{(1)} = \frac{1}{5}(3 - \frac{2}{3}) = \frac{7}{15}$

$z^{(1)} = \frac{1}{3}(-2 - \frac{14}{15}) = -\frac{44}{45}$

$x^{(2)} = \frac{1}{3}(1 - \frac{7}{15} + \frac{44}{45}) \approx 0.5037$

$y^{(2)} = \frac{1}{5}(3 - 2(0.5037) - 2(-\frac{44}{45})) \approx 0.78963$

$z^{(2)} = \frac{1}{3}(-2 - 2(0.78963)) \approx -1.193087$

2. $\underline{x}^{(k+1)} = \underline{P}^{-1}\underline{N}\,\underline{x}^{(k)} + \underline{P}^{-1}(\underline{r}^{(k)} + \underline{A}\,\underline{x}^{(k)})$ (Since $\underline{r}^{(k)} = \underline{b} - \underline{A}\,\underline{x}^{(k)}$)

$= \underline{P}^{-1}(\underline{N} + \underline{A})\underline{x}^{(k)} + \underline{P}^{-1}\underline{r}^{(k)}$

$= \underline{P}^{-1}\underline{P}\,\underline{x}^{(k)} + \underline{P}^{-1}\underline{r}^{(k)}$

$= \underline{x}^{(k)} + \underline{P}^{-1}\underline{r}^{(k)}$

3. $k_1 = .05(2(0)(2)+1) = .05$

$k_2 = .05\left(2\left(\frac{.05}{2}\right)\left(2+\frac{.05}{2}\right)+1\right) = 0.055063$

$k_3 = .05\left(2\left(\frac{.05}{2}\right)\left(2+\frac{.055063}{2}\right)+1\right) = .055069$

$k_4 = .05(2(.05)(2+.055069)+1) = .060275$

$y_1 = 2 + \frac{1}{6}(.05 + 2(.055063) + 2(.055069) + .060275)$

$= 2.05509$

$k_1 = .05(2(.05)(2.05509)+1) = .060275$

$k_2 = .05\left(2\left(.05+\frac{.05}{2}\right)\left(2.05509+\frac{.060275}{2}\right)+1\right) = .065639$

$k_3 = .05\left(2\left(.05+\frac{.05}{2}\right)\left(2.05509+\frac{.065635}{2}\right)+1\right) = .065659$

$k_4 = .05(2(.1)(2.05509+.065659)+1) = .071207$

$y_2 = 2.05509 + \frac{1}{6}(0.060275 + 2(0.065639) + 2(0.065659) + 0.071207)$

$= 2.25213$

4. ⓐ $\tau_{n+1} = \frac{1}{h}\left[y(t_{n+1}) - \left\{y_n + \frac{23}{12}hy_n' - \frac{16}{12}hy_{n-1}' + \frac{5}{12}hy_{n-2}'\right\}\right], \quad y_n = y(t_n)$

$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \frac{h^3}{3!}y'''(t_n) + \frac{h^4}{4!}y^{(4)}(\xi)$

$y_{n-1}' = y'(t_{n-1}) = y'(t_n) - hy''(t_n) + \frac{(-h)^2}{2}y'''(t_n) + \frac{(-h)^3}{3!}y^{(4)}(\xi_1)$

$y_{n-2}' = y'(t_{n-2}) = y'(t_n) - 2hy''(t_n) + \frac{(-2h)^2}{2}y'''(t_n) + \frac{(-2h)^3}{3!}y^{(4)}(\xi_2)$

$\tau_{n+1} = \frac{1}{h}\Big[hy'(t_n)\overbrace{\left\{1 - \frac{23}{12} + \frac{16}{12} - \frac{5}{12}\right\}}^{0} + h^2y''(t_n)\overbrace{\left\{\frac{1}{2} - \frac{16}{12} + \frac{10}{12}\right\}}^{0}$

$+ h^3y'''(t_n)\overbrace{\left\{\frac{1}{6} + \frac{16}{24} - \frac{10}{12}\right\}}^{0} + h^4y^{(4)}(t_n)\left\{\frac{1}{24} - \frac{16}{12}\left(\frac{1}{6}\right) + \frac{5}{12}\left(\frac{8}{6}\right)\right\}$

$+ O(h^4)\Big]$

$= \frac{3}{8}h^3y^{(4)}(\xi)$

ⓑ Since $\tau_{n+1} = O(h)$ it is consistent

$P(\lambda) = \lambda^3 - 1 = 0$

$\Rightarrow (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$

$\lambda = 1, \ \dfrac{-1 \pm \sqrt{1-4}}{2}$

$\left| -\frac{1}{2} + \frac{\sqrt{3}}{2} i \right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\left| -\frac{1}{2} - \frac{\sqrt{3}}{2} i \right| = 1$

$\Rightarrow \ |\lambda_i| = 1, 1, 1$ satisfies root condition

weakly stable since repeated.

Thm 7.24: consistent + root condition $\Rightarrow$ converges