# MAT 333 SECTIONS 001 AND 002 SPRING 2001

## Graded Homework for Chapters 4 and 5

4.1 The general form of a linear, homogeneous, second-order equation with constant coefficients is

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

Write the first-order system for this equation, and write this system in matrix form.

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For the next three problems, 5.1, 5.2, and 5.3, do the following three parts:

(a) Solve the system of two equations.

(b) Graph both equations on the same graph, and indicate the solution(s) (if any) on this graph.

(c) Write the system in the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and find the determinant of $\mathbf{A}$.

Then answer the question: Under what conditions does there exist a unique solution to $\mathbf{Ax} = \mathbf{b}$?

5.1

$$\begin{aligned} 2x + y &= 1 \\ 3x - 2y &= 5. \end{aligned}$$

5.2

$$\begin{aligned} \frac{1}{2}x - 3y &= 8 \\ \frac{3}{2}x - 9y &= -2. \end{aligned}$$

5.3

$$\begin{aligned} 2x + 3y &= 0 \\ 6x + 9y &= 0. \end{aligned}$$

5.4 Verify that

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

Note that you do not have to solve for the inverse.

5.5 Find the inverse of

$$\begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}.$$

5.6 Use elementary row operations to solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 8 \\ 3x_1 - x_3 &= 3. \end{aligned}$$

5.7 Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

5.8 Show that the general solution of

$$\mathbf{x}' = \mathbf{A(t)x} + \mathbf{f(t)}$$

is the sum of any particular solution $\mathbf{x_p}$ of this equation and the general solution $\mathbf{x_c}$ of the corresponding homogeneous equation.

5.9 Consider a two-mass three spring system, with no external forces, whose equations of motion are given by

$$x_1'' = -2x_1 + x_2 \quad x_2'' = x_1 - 2x_2.$$

(a) Transform the system into a system of four first order equations by letting $y_1 = x_1$, $y_2 = x_1'$, $y_3 = x_2$, and $y_4 = x_2'$.

(b) Find the eigenvalues of the coefficient matrix for the system in part (a).

(c) Solve the system in part (a) subject to the initial conditions $\mathbf{y^T(0)} = [\mathbf{2, 1, 2, 1}]$. Describe the physical motion of the spring-mass system that corresponds to this solution.

(d) Solve the system in part (a) subject to the initial conditions $\mathbf{y^T(0)} = [\mathbf{2}, \sqrt{\mathbf{3}}, -\mathbf{2}, -\sqrt{\mathbf{3}}]$. Describe the physical motion of the spring-mass system that corresponds to this solution.

(e) Observe that the spring-mass system has two natural modes of oscillation in this case. How are the natural frequencies related to the eigenvalues of the coefficient matrix? Do you think that there might be a third natural model of oscillation with a different frequency?

5.10 Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} -3/2 & 1 \\ -1/4 & -1/2 \end{bmatrix} \mathbf{x},$$

and describe how the solutions behave as $t \to \infty$.

5.11 Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix}. \tag{1}$$

(a) Show that $\lambda = 1$ is a triple eigenvalue of defect 1, and that we may take the two linearly independent eigenvectors as

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}.$$

Find two linearly independent solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$.

(b) To find a third solution assume that $\mathbf{x} = \mathbf{v}te^t + \mathbf{w}e^t$; then show that $\mathbf{v}$ and $\mathbf{w}$ must satisfy

$$(\mathbf{A} - \mathbf{I})\,\mathbf{v} = 0, \tag{2}$$

$$(\mathbf{A} - \mathbf{I})\,\mathbf{w} = \mathbf{v}. \tag{3}$$

(c) Show that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where $c_1$ and $c_2$ are arbitrary constants, is the most general solution of (2). Show that in order to solve (3) it is necessary that $c_1 = c_2$.

(d) It is convenient to choose $c_1 = c_2 = 2$. For this choice, find $\mathbf{v}$, $\mathbf{w}$, and a third linearly independent solution $\mathbf{x}_3(t)$.