$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}$$

$$\frac{-2x+4}{\cancel{(x^2+1)(x-1)^2}} = \left[\frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}\right](x^2+1)(x-1)^2$$

$$-2x+4 = Ax+B(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$-2x+4 = Ax(x^2-2x+1)+B(x^2-2x+1) + C(x^3-x^2-x-1) + D(x^2+1)$$

$$-2x+4 = Ax^3-2Ax^2+Ax+Bx^2-2Bx+B+Cx^3-Cx^2+Cx-C+Dx^2+D$$

$$-2x+4 = x^3(A+C) + x^2(B-2A-C+D) + x(A-2B+C) + (B-C+D)$$

$(A+C)=0$ ④ $A=-C$, $\boxed{A=2}$

$(B-2A-C+D)=0$ ③ $1+2C-C+C+3=0$, $2C=-4$, $\boxed{C=-2}$

$(A-2B+C)=-2$ ① $-C-2B+C=-2$, $-2B=-2$, $\boxed{B=1}$

$(B+D-C)=4$ ② $1+D-C=4$, $3=D-C$ $D=3+C$

⑤ $1+D+2=4$, $\boxed{D=1}$

$$\int \frac{2x+1}{x^2+1} + \int \frac{-2}{x-1} + \int \frac{1}{(x-1)^2}$$

$$= \int \frac{2x}{x^2+1} + \int \frac{1}{x^2+1} + \int \frac{-2}{x-1} + \int \frac{1}{(x-1)^2}$$

$$= \ln|x^2+1| + \arctan x - 2\ln|x-1| - (x-1)^{-1} + C$$

$$= \ln\left|\frac{x^2+1}{(x-1)^2}\right| + \arctan x - (x-1)^{-1} + C$$

Extra credit

$$\int \frac{x^4}{x^4-1}\,dx$$

$$\int \frac{x^4}{(x^2+1)(x^2-1)}\,dx$$

$$\left[\frac{x^4}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}\right](x^2+1)(x+1)(x-1)$$

$$x^4 = (Ax+B)(x^2-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)$$

$$x^4 = Ax^3 - Ax + Bx^2 - B + Cx^3 - Cx^2 + Cx - C + Dx^3 + Dx^2 + Dx + D$$

Coefficient of $1 \cdot x^4 = 1$
Coefficient of x^3 : $A + C + D = 0$
Coefficient of x^2 : $B - C + D = 0$
Coefficient of x^1 : $-A - B + C + D = 0$
Coefficient of x^0 : $-B - C + D = 1$

$$x^4 - 1 \overline{)\, x^4} \quad 1$$
$$\underline{-(x^4 - 1)}$$
$$1$$

$$x^4 - 1 = (x^2)^2 - 1^2 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$

$$\left[\frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}\right](x^2+1)(x+1)(x-1)$$

$$1 = A(x^2+1)(x-1) + B(x+1)(x^2+1) + (Cx+1)(x+1)(x-1)$$

$x = 1$
$1 = B(x+1)(x^2+1)$
$1 = B(x^3+x^2+x+1)$
$1 = B(1+1+1+1)$
$1 = 4B$
$B = \frac{1}{4}$

$x = -1$
$1 = A(1+1)(-2)$
$1 = -4A$
$A = -\frac{1}{4}$

$$1 = A(x^2+1)(x-1) + B(x^2+1)(x+1) + (Cx+D)(x^2-1)$$

$$1 = A(x^3 - x^2 + x - 1) + B(x^3 + x^2 + x + 1)$$
$$+ Cx^3 - Cx + Dx^2 - D$$

$$1 = Ax^3 - Ax^2 + Ax - A + Bx^3 + Bx^2 + Bx + B$$
$$+ Cx^3 - Cx + Dx^2 - D$$

$$1 = (A+B+C)x^3 + (B-A+D)x^2 + (A+B-C)x + (B-A-D)$$

$A+B+C=0$ $\qquad -\frac{1}{4} + \frac{1}{4} + C = 0 \Rightarrow C = 0$

$B-A+D=0$

$A+B-C=0$ $\qquad -\frac{1}{4} - \left(\frac{1}{4}\right) + D = 0 \Rightarrow D = -\frac{2}{8} = -\frac{1}{2}$

$B-A-D=1$

$$-\frac{1}{4} - \left(\frac{1}{4}\right) - \left(-\frac{1}{2}\right) = 1$$

$$-\frac{1}{4}\int \frac{1}{x+1}\,dx = -\frac{1}{4}\ln|x+1|$$

$$\frac{1}{4}\int \frac{1}{x-1}\,dx = \frac{1}{4}\ln|x-1|$$

$$\frac{1}{2}\int \frac{1}{x^2+1}\,dx = \frac{1}{2}\tan^{-1}x$$

$$\int \frac{x^4}{x^4-1}\,dx = x + \frac{1}{4}\left[\ln|x-1| - \ln|x+1|\right]$$
$$-\frac{1}{2}\tan^{-1}x + C$$

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx$$

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$x^4+1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x$$

$$= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$

$$x^4+1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

set $x=0 \rightarrow A=1$ and $C=0$

$E=0$

$A+B=1$ $\qquad 2A+B+D=0$

$B=1-A$ $\qquad 2-0+D=0$

$0=B$ $\qquad D=-2$

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx = \int\left(\frac{1}{x} + \frac{0+0}{x^2+1} + \frac{-2x+0}{(x^2+1)^2}\right)dx$$

$$\int\left[\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right]dx$$

$$\int \frac{1}{x}\,dx - \int \frac{2x}{(x^2+1)^2}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\int \frac{1}{x}\,dx - \int \frac{du}{(u)^2}$$

$$\ln(x) + \frac{1}{u}$$

$$\boxed{\ln(x) + \frac{1}{(x^2+1)} + K}$$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$$

$(x-1)(x-1)$

$(x^2-2x+1)(x+1)$

$x^3-2x^2+x+x^2-2x+1$

x^3-x^2-x+1

$$\begin{array}{r}
x+1 \\
x^3-x^2-x+1 \,\overline{)\, x^4-2x^2+4x+1} \\
-(x^4-x^3-x^2+x) \\
\hline
0+x^3-x^2+3x+1 \\
-x^3+x^2-x-1 \\
\hline
4x
\end{array}$$

$$\int x+1+\frac{4x}{x^3-x^2-x+1}\,dx$$

$$\int\frac{4x}{x^3-x^2-x+1}=\int\frac{4x}{(x-1)^2(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$$

$A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$A(x^2-1)+B(x+1)+C(x^2-2x+1)$

$A+C=0$ $\quad C=-A$ $\quad A+B-A=0$

$B-2C=4$ $\quad -2A+B=0$

$-A+B+C=0$ $\quad B=2A$

$A=1,\ C=-1,\ B=2$ $\quad 2A-(-2A)=4$

$4A=4$

$A=1$

$$\int x+1+\frac{1}{(x-1)}+\frac{2}{(x-1)^2}-\frac{1}{(x+1)}\,dx$$

$$\frac{x^2}{2}+x+\ln(x-1)-\frac{2}{(x-1)}-\ln(x+1)$$

$$\left[\frac{x^2}{2}+x+\ln\left|\frac{(x-1)}{(x+1)}\right|-\frac{2}{(x-1)}\right]$$

DATE

$$\int \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$2x+4 = A(x^2)(x-2) + B(x)(x-2) + C(x^2)$$

$$2x+4 = Ax^3 - 2Ax^2 + Bx^2 - 2Bx + Cx^3$$

$$2x+4 = x^2(B-2A) + x^3(A+C) - 2Bx$$

$$2x+4 = -2Bx$$

$$-x-2 = Bx$$

$$-2 = Bx + x$$

$$-2 = x(B+1)$$

$$\frac{-2}{x} = B+1$$

$$\frac{-2}{x} - 1 = B$$

$$\frac{-2-x}{x} = B$$

$$2y+4 = -2\left(\frac{-2-x}{x}\right)x$$

$$2x+4 = \frac{4x+2x^2}{x} ; \frac{\cancel{x}(4+2x)}{\cancel{x}}$$

$$2x+4 = 4+2x$$

$$\boxed{x=1}$$

$$\Rightarrow$$

$0 = B - 2A$

$0 = \frac{-2-x}{x} - 2A$

$\frac{2+x}{x} = -2A$

$\boxed{\frac{-3}{2} = A}$

$0 = A + C$

$-A = C$

$\frac{3}{2} = C$

$$\int \frac{-3/2}{x} + \frac{\frac{-2-x}{x}}{x^2} + \frac{3/2}{(x-2)}\,dx$$

$$\Rightarrow -3/2\left[\ln|x|\right] + \int \frac{-2-x}{x^3}\,dx + \frac{3}{2}\left[\ln|x-2|\right] + C$$