

Phase and Amplitude Errors in a Modified Chebyshev Method

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Introduction

The Chebyshev pseudospectral method for derivative approximations can be made more efficient by using the grid transformation introduced by Kosloff and Tal-Ezer [4]. Specifically, using the Chebyshev pseudospectral method, we may write the derivative approximation as a matrix-vector product

$$u' = Du.$$

The method in [4] maps the grid at the Chebyshev points x according to

$$y(x) = g(x) = \frac{\sin^{-1}(\alpha x)}{\sin^{-1}(\alpha)}, \quad 0 < \alpha < 1. \quad (1)$$

The derivative on the transformed grid may be written

$$u' = ADu,$$

where A has elements

$$A_{kk} = \frac{1}{g'(\alpha, x_k)} = \frac{\sin^{-1}(\alpha) \sqrt{1 - \alpha^2 x_k^2}}{\alpha}.$$

We see that A damps the large elements of D as α tends to 1. This allows larger time steps in the solution of partial differential equations. However, there is a singularity at $\alpha = 1$. In this work we study the phase and amplitude errors of the mapped method for various α .

Phase and Amplitude Errors

In [3], Kopriva determines that the phase and amplitude errors for the Chebyshev pseudospectral method decay exponentially as the number of points per wavelength increases. We have applied the same technique to calculate phase and amplitude errors for the modified method. Specifically we solve the 1-D wave equation

$$u_t + u_x = 0 \quad -1 \leq x \leq 1, 0 < t < 8 \quad (2)$$

$$u(x, 0) = e^{ik\pi x} \quad -1 \leq x \leq 1 \quad (3)$$

$$u(-1, t) = e^{-ik\pi(1+t)} \quad 0 < t < 8, \quad (4)$$

with time stepping by the fourth order Runge-Kutta method. The time dependent boundary conditions were imposed so as to keep the formal accuracy of the Runge-Kutta method [2].

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The spatial derivative was approximated using a matrix-vector multiply as described in the Introduction, and also with a discrete fast Fourier transform. The approach described in [1] for the Chebyshev method immediately extends for the mapped method. We found that our conclusions are valid regardless of the method for the implementation of the derivative.

The phase and amplitude errors for the mapped method decay exponentially as the number of points per wavelength increases if the choice of α depends on the grid size N . Specifically, the choices of α for which this is true are those suggested in [4] where either accuracy or resolution is optimal. However, for values of α that can achieve single precision accuracy (not all α can), the phase and amplitude errors are less than those from the Chebyshev method when there are less than seven points per wavelength.

The phase and amplitude errors for the second derivative matrices were also calculated. Here we considered the one-dimensional two way wave equation,

$$u_{tt} = u_{xx} \quad -1 \leq x \leq 1, \quad 0 \leq t < 8, \quad (5)$$

$$\left. \begin{aligned} u(x, 0) &= e^{ik\pi x} \\ u_t(x, 0) &= -ik\pi e^{ik\pi x} \end{aligned} \right\} \quad -1 \leq x \leq 1 \quad (6)$$

$$\left. \begin{aligned} u(-1, t) &= e^{-ik\pi(1+t)} \\ u(1, t) &= e^{ik\pi(1-t)} \end{aligned} \right\} \quad 0 \leq t < 8, \quad (7)$$

for which $u(x, t) = e^{ik\pi(x-t)}$. The method to impose the time dependent boundary conditions was extended to the second order derivative approximation, as was the discrete fast Fourier transform. Again we see that irregardless of the implementation, the phase and amplitude errors decay exponentially only if α depends on N . But for a small number of points per wavelength, values of α that can achieve single precision accuracy, have smaller phase and amplitude errors.

Bibliography

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