Strong measure zero sets, filters, and pointwise convergence.

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The notion of a strong measure zero space (but not this terminology) was introduced in [1] by E. Borel. A separable metric space X has strong measure zero if there is for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers a partition $X = \cup_n X_n$ such that for each n the diameter of X_n is less than ϵ_n. In [20] we found characterizations of the notion of a strong measure zero set in terms of certain selection principles applied to open covers, and also in terms of Ramseyan partition relations for certain families of open covers. Also, the property of having strong measure zero in all finite powers was characterized like this. Due to successes in using the concept of a filter on the set of natural numbers to describe certain covering properties of sets of real numbers, as in [10], and of using the closure properties of function spaces, as in [17], the question arose whether these methods would also yield analogous descriptions of strong measure zero spaces. In the paper [9] we made a beginning in this direction by showing that for certain function spaces derived from a space X, the strong measure zero-ness of all finite powers of X is equivalent to certain closure properties of the associated function space. The function space associated with X was obtained by first associating with X a subspace $T(X)$ of the Alexandroff double of the closed unit interval [0,1], and by then taking the function space $\mathsf{C}_s(T(X))$ – this space differs from the usual space topologized by the topology of pointwise convergence in that we consider a coarser topology described in terms of a dense discrete subspace consisting of the isolated points of $T(X)$.

We shall see below that the intermediate construction $T(X)$ can be avoided, and that the traditional topology of pointwise convergence is sufficient for the task of detecting the strong measure zero ness of the domain space. We shall also see that the filter description can be carried through here.

To set the stage for the work to be done, we first introduce some necessary notation and concepts, and cite one of the principal results from [20] which will be used here. An open cover $\mathcal{U}$ of a space is an ω–cover if the space itself is not a member of $\mathcal{U}$, but for each finite subset F of the space there is a $U \in \mathcal{U}$ with $F \subseteq U$. Thus, let Y be a compact metric space, and let X be a subspace of Y.

$\mathcal{O}$: The collection of all open covers of Y;

$\mathcal{D}$: The collection of all covers of X by sets open in Y;

*Supported by NSF grant DMS 95-05375

Ω: The collection of ω–covers of Y;

$\mathcal{D}_\Omega$: The collection of ω–covers of X by sets open in Y.

For $\mathcal{A}$ and $\mathcal{B}$ subsets of an infinite set S, the symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n we have $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is in $\mathcal{B}$. The game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ associated with $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is defined as follows: Players ONE and TWO play an inning per positive integer. In the n–th inning ONE first chooses an $O_n \in \mathcal{A}$, to which TWO responds by choosing $T_n \in O_n$. A play

$$O_1, T_1, \dots, O_n, T_n, \dots$$

is won by TWO if $\{T_n : n \in \mathbb{N}\} \in \mathcal{B}$, and otherwise ONE wins.

For each k the symbol

$$\mathcal{A} \to (\mathcal{B})^2_k$$

means: For each $A \in \mathcal{A}$, and for each function $f : [A]^2 \to \{1, \dots, k\}$ there is a subset B of A and an $i \in \{1, \dots, k\}$ such that $B \in \mathcal{B}$, and f is constant, of value i, on $[B]^2$.

The following theorem is one of the principal results of [20] which was alluded to above.

Theorem 1 ([20]) *For X a subspace of a σ–compact metric space Y, the following are equivalent:*

1. *Each finite power of X has strong measure zero;*
2. *$\mathsf{S}_1(\Omega, \mathcal{D}_\Omega)$ holds for Y;*
3. *ONE has no winning strategy in the game $\mathsf{G}_1(\Omega, \mathcal{D}_\Omega)$;*
4. *For each k the relation $\Omega \to (\mathcal{D}_\Omega)^2_k$ holds.*

1 Filters

In [10] an operation was described for associating with a countable ω–cover $\mathcal{U}$ of a set X of real numbers a filter $\mathcal{H}(\mathcal{U})$ on the set of positive integers, and conversely, for associating with a filter $\mathcal{F}$ on the set of positive integers a set X of real numbers, and a countable ω–cover $\mathcal{U}$ of X, such that $\mathcal{F} = \mathcal{H}(\mathcal{U})$. This operation was then used to translate back-and-forth between the theories of filters and of open covers. We now apply the process here to give characterizations of being strong measure zero in all finite powers in terms of properties of $\mathcal{H}(\mathcal{U})$.

To begin, let $\mathcal{U}$ be a countable open cover of [0,1], say $(U_n : n \in \mathbb{N})$ enumerates $\mathcal{U}$ bijectively. For each finite subset F of [0,1], define

$$A_F = \{n : F \subseteq U_n\}.$$

Then $\mathcal{H}(\mathcal{U})$ is the filter generated by the family $\{A_F : F \subset [0,1] \text{ finite}\}$. Let $\mathcal{I}(\mathcal{U})$ be the filter generated by the family $\{A_F : F \subset X \text{ finite}\}$.

For a family $\mathcal{A}$ of subsets of $\mathbb{N}$ we define:

$$\mathcal{A}^+ := \{B \subseteq \mathbb{N} : (\forall A \in \mathcal{A})(A \cap B \neq \emptyset)\}.$$

Since $\mathcal{I}(\mathcal{U})$ is a subset of $\mathcal{H}(\mathcal{U})$, we have $\mathcal{U}(\mathcal{U})^+ \subseteq \mathcal{I}(\mathcal{U})^+$.

For families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{N}$ the symbol $\mathsf{Q}^+(\mathcal{A},\mathcal{B})$ denotes the statement:

> For each element $A \in \mathcal{A}^+$, and for each partition $A = \cup_{n\in\mathbb{N}} I_n$ of A into pairwise disjoint finite sets, there is an element B of $\mathcal{B}^+$ such that for each n, $I_n \cap B$ has at most one element, and $B \subseteq A$.

Theorem 2 *For a subset X of [0,1] the following are equivalent:*

1. *Each finite power of X has strong measure zero.*
2. *For each countable ω–cover $\mathcal{U}$ of [0,1] the statement $\mathsf{Q}^+(\mathcal{H}(\mathcal{U}),\mathcal{I}(\mathcal{U}))$ holds.*

Proof: For the proof that 1 implies 2 we recall from the principal theorem cited above that since X has strong measure zero in all finite powers, ONE has no winning strategy in the game $\mathsf{G}_1(\Omega,\mathcal{D}_\Omega)$, played on [0,1]. This is used as follows: Let $\mathcal{U}$, enumerated bijectively as $(U_n : n \in \mathbb{N})$, be an ω–cover of [0,1]. Let A be an element of $\mathcal{H}(\mathcal{U})^+$, and let $(I_n : n \in \mathbb{N})$ be a pairwise disjoint sequence of finite subsets of A with union equal to A. Then $\mathcal{V} = \{U_n : n \in A\}$ is an ω–cover of [0,1] and, setting $\mathcal{V}_n = \{U_m : m \in I_n\}$, each $\mathcal{V}_n$ is a finite set such that $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{V}_n$.

Consider the following strategy, F, for ONE in the game $\mathsf{G}_1(\Omega,\mathcal{D}_\Omega)$: In the first inning ONE plays $F(\emptyset) = \mathcal{V}$. If TWO responds with $T_1 \in F(\emptyset)$, then ONE chooses n_1 so that $T_1 \in \mathcal{V}_{n_1}$, and plays $F(T_1) = \cup_{n>n_1}\mathcal{V}_n$. If TWO now chooses $T_2 \in F(T_1)$, then ONE chooses n_2 such that $T_2 \in \mathcal{V}_{n_2}$, and plays $F(T_1,T_2) = \cup_{n>n_2}\mathcal{V}_n$, and so on. Since F is not a winning strategy for ONE in $\mathsf{G}_1(\Omega,\mathcal{D}_\Omega)$, consider a play which is lost by ONE, say $F(\emptyset), T_1, F(T_1), T_2, F(T_1,T_2), \ldots$. Then $\{T_m : m \in \mathbb{N}\}$ is an element of $\mathcal{D}_\Omega$. For each m choose j_m such that $T_m = U_{j_m}$. Then the set $B = \{j_m : m \in \mathbb{N}\}$ is an element of $\mathcal{I}(\mathcal{U})^+$, and it meets each I_n in at most one point.

Next we prove that 2 implies 1 by showing that 2 implies that [0,1] satisfies the partition relation $\Omega \to (\mathcal{D}_\Omega)^2_k$ for each k. Thus, let a k and an ω–cover $\mathcal{U}$ of [0,1], and a coloring $f : [\mathcal{U}]^2 \to \{1,2,\ldots,k\}$ be given. Enumerate $\mathcal{U}$ bijectively as $(U_n : n \in \mathbb{N})$. Since [0,1] has the property $\mathsf{S}_{fin}(\Omega,\Omega)$, it satisfies the partition relation $\Omega \to \lceil\Omega\rceil^2_k$. Thus, let $\mathcal{V} \subset \mathcal{U}$ be an ω–cover of [0,1], and let $\mathcal{V}_m,\ m \in \mathbb{N}$ be a sequence of pairwise disjoint finite subsets of $\mathcal{V}$, and let $i \in \{1,\ldots,k\}$ be given such that whenever $m \neq n$ and whenever $U \in \mathcal{V}_m$ and $V \in \mathcal{V}_n$, then $f(\{U,V\}) = i$. Apply $\mathsf{Q}^+(\mathcal{H}(\mathcal{U}),\mathcal{I}(\mathcal{U}))$: Since $\mathcal{V}$ is an ω–cover of X we have $A = \{n : U_n \in \mathcal{V}\} \in \mathcal{H}(\mathcal{U})^+$. For each m, $I_m = \{n : U_n \in \mathcal{V}_m\}$ is finite, and the sequence of I_n's is a partition of A. Thus, choose a subset B of A such that for each n $B \cap I_n$ has at most one element, and $B \in \mathcal{H}(\mathcal{I})^+$. But then the set $\{U_n : n \in B\}$ is in $\mathcal{D}_\Omega$, and meets each $\mathcal{V}_n$ in exactly one point; thus on the set of pairs from this set f is constant of value i.

2 Pointwise convergence.

Since translations of strong measure zero sets are strong measure zero, and since the union of countably many strong measure zero sets is strong measure zero, a set X of real numbers is strong measure zero if, and only if, $X+\mathbb{Q}$ is strong measure zero. We may therefore, when determining which sets of real numbers are of strong measure zero, confine our attention to dense sets. For the remainder of this section we shall assume that X is a dense subset of [0,1]. We define two topologies on C([0,1]), as follows: The one topology is the usual topology of pointwise convergence – endowed with this topology this space is denoted C_p([0,1]). For the second topology, we take for each open subset U of $\mathbb{R}$, and each finite subset F of X, the set V[F,U] to be $\{f \in \mathrm{C}[0,1]) : f[F] \subset U\}$. Then we take the topology generated on C([0,1]) by the family $\{V[F,U] : F \subset X$ finite and $U \subset \mathbb{R}$ open $\}$. We denote C([0,1]) endowed with this topology by the symbol C_p^X([0,1]).

The symbol $\underline{0}$ denotes the function which is everywhere equal to zero. We define the following two notions:

$$\Omega^1_{\underline{0}} = \{B \subset \mathrm{C}([0,1]) \setminus \{\underline{0}\} : \underline{0} \text{ in the } \mathsf{C}_p\text{–closure of } B\}$$

and

$$\Omega^2_{\underline{0}} = \{B \subset \mathrm{C}([0,1]) \setminus \{\underline{0}\} : \underline{0} \text{ in the } \mathsf{C}_p^X \text{ – closure of } B\}.$$

Theorem 3 *For X a dense subset of [0,1], the following are equivalent:*

1. *C([0,1]) satisfies the selection principle $\mathsf{S}_1(\Omega^1_{\underline{0}}, \Omega^2_{\underline{0}})$.*
2. *Each finite power of X has strong measure zero.*
3. *ONE has no winning strategy in the game $\mathsf{G}_1(\Omega^1_{\underline{0}}, \Omega^2_{\underline{0}})$.*
4. *For each k, C([0,1]) satisfies the partition relation $\Omega^1_{\underline{0}} \to (\Omega^2_{\underline{0}})^2_k$.*

Proof: That 1 implies 2: For each n let $(U^n_k : k \in \mathbb{N})$ bijectively enumerate an open ω–cover of [0,1]. Write $U^n_k = \cup_{j\in\mathbb{N}} U^n_{k,j}$ where for each j, $U^n_{k,j} \subset U^n_{k,j+1}$ are closed sets. By Urysohn's Lemma we find for each n, k, j a continuous function $f^n_{k,j}$ on [0,1] which has value zero on $U^n_{k,j}$, and value one outside U^n_k. Observe that for each n the set $B_n = \{f^n_{k,j} : k, j \in \mathbb{N}\}$ is an element of $\Omega^1_{\underline{0}}$. Applying $\mathsf{S}_1(\Omega^1_{\underline{0}}, \Omega^2_{\underline{0}})$, we find for each n a pair k_n, j_n such that $\{f^n_{k_n,j_n} : n \in \mathbb{N}\}$ is in $\Omega^2_{\underline{0}}$. Consider the sequence $(U^n_{k_n} : n \in \mathbb{N})$. For F a finite subset of X there exists an n such that $f^n_{k_n,j_n}[F] \subset (-\frac{1}{2}, \frac{1}{2})$, and so $F \subset U^n_{k_n}$. Thus, $\{U^n_{k_n} : n \in \mathbb{N}\}$ is an element of $\mathcal{D}_\Omega$. We have shown that [0,1] has the property $\mathsf{S}_1(\Omega, \mathcal{D}_\Omega)$, and thus that each finite power of X has strong measure zero.

To prove that 2 implies 3, we proceed as follows: We associate with a strategy F of ONE of the game $\mathsf{G}_1(\Omega^1_{\underline{0}}, \Omega^2_{\underline{0}})$ played on C([0,1]) a strategy G of ONE of the game $\mathsf{G}_1(\Omega, \mathcal{D}_\Omega)$ played on [0,1]. This is done as follows:

Let $F(\emptyset)$ be $(f_n : n \in \mathbb{N})$. (We are using the fact that C([0,1]) has countable tightness to justify the fact that we may confine ourselves to countable members of $\Omega^1_{\underline{0}}$.) Since no f_n is exactly zero, and since X is dense in [0,1], there is for each n an m_n such that for all $k \geq m_n$ the set $\{y : |f_n(y)| < \frac{1}{k}\}$ does not cover X. Define:

$$G(\emptyset) := \{f_n^{\leftarrow}[(-\frac{1}{m}, \frac{1}{m})] : n \in \mathbb{N}, m \geq m_n\}.$$

Then $G(\emptyset)$ is an ω–cover of X. Suppose that TWO responds by choosing a member T_1 of $G(\emptyset)$. Then T_1 is of the form $f_{n_1}^{\leftarrow}[(-\frac{1}{k_1}, \frac{1}{k_1})]$, and we assign to TWO of the game on C([0,1]) the move $S_1 = f_{n_1}$.

ONE now responds with $F(f_{n_1}) = (f_{n_1,m} : m \in \mathbb{N})$, a bijectively enumerated member of $\Omega^1_{\underline{0}}$. Using this and the fact that there is for each n an $m_{n_1,n} > m_{n_1} + k_1$ such that for $k \geq m_{n_1,n}$ the set X is not covered by $f_{n_1,n}^{\leftarrow}[(-\frac{1}{k}, \frac{1}{k})]$, we define

$$G(T_1) := \{f_{n_1,n}^{\leftarrow}[(-\frac{1}{k}, \frac{1}{k})] : n \in \mathbb{N}, k \geq m_{n_1,n}\}.$$

Suppose TWO responds to $G(T_1)$ by choosing a member T_2 of it. Now T_2 is of the form $\{y : |f_{n_1,n_2}(y)| < \frac{1}{k_2}\}$, where $k_2 \geq m_{n_1,n_2} > m_{n_1} + k_1$. Assign to TWO of the game on C([0,1]) the move $S_2 = f_{n_1,n_2}$, and so on.

Since each finite power of X has strong measure zero, G is not a winning strategy for ONE in the game $\mathsf{G}_1(\Omega, \mathcal{D}_\Omega)$. Consider a play

$$G(\emptyset), T_1, G(T_1), T_2, G(T_1, T_2), \ldots$$

which is won by TWO. Then $\{T_n : n \in \mathbb{N}\}$ is a member of $\mathcal{D}_\Omega$. Moreover, a recursive computation shows that there are sequences $n_1,\ n_2,\ \ldots$ and $k_1,\ k_2,\ \ldots$ such that $k_1 < k_2 < \ldots$, and for each j we have T_j is the set $\{y : |f_{n_1,\ldots,n_j}(y)| < \frac{1}{k_j}\}$. But then it follows that $\{f_{n_1,\ldots,n_j} : j \in \mathbb{N}\}$ is an element of $\Omega^2_{\underline{0}}$, and so the play

$$F(\emptyset), f_{n_1}, F(f_{n_1}), f_{n_1,n_2}, F(f_{n_1}, f_{n_1,n_2}), \ldots$$

of $\mathsf{G}_1(\Omega^1_{\underline{0}}, \Omega^2_{\underline{0}})$ on C([0,1]) is lost by ONE.

The proof that 2 implies 3 uses a standard argument which we include here for the convenience of the reader: Let an element A of $\Omega^1_{\underline{0}}$ be given, and let a function $f : [A]^2 \rightarrow \{1, \ldots, k\}$ be given. We may assume that A is countable, and enumerate it bijectively as $(f_n : n \in \mathbb{N})$. Recursively construct sequences $(A_n : n \in \mathbb{N})$ and $(i_n : n \in \mathbb{N})$ such that each A_n is a member of $\Omega^1_{\underline{0}}$ and each i_n is a member of $\{1, 2, \ldots, k\}$, and:

1. $A_1 = \{f_n : n > 1 \text{ and } f(\{f_1, f_n\}) = i_1\}$;
2. $A_{n+1} = \{f_m \in A_n : m > n+1 \text{ and } f(\{f_{n+1}, f_m\}) = i_{m+1}\}$.

For each j in $\{1, \ldots, k\}$ define $\mathcal{E}_j = \{f_n : i_n = j\}$. Then for each n there is a j_n such that $A_n \cap \mathcal{E}_{j_n}$ is in $\Omega^1_{\underline{0}}$. Since the sequence of A_n's is decreasing, we may assume that for all n we have $j_n = j$. We now consider the sequence $(A_n \cap \mathcal{E}_j : f_n \in \mathcal{E}_j)$.

Consider the following strategy, F, for ONE of the game $\mathsf{G}_1(\Omega^1_{\underline{0}}, \Omega^2_{\underline{0}})$. With m_1 the minimal n such that $f_n \in \mathcal{E}_j$, ONE's first move is $F(\emptyset) = A_{m_1} \cap \mathcal{E}_j$.

Should TWO respond with $f_{m_2} \in F(\emptyset)$, then ONE plays $F(f_{m_2}) = A_{m_2} \cap \mathcal{E}_j$ (notice that $m_1 < m_2$). If TWO responds with $f_{m_3} \in F(f_{m_2})$, then ONE plays $F(f_{m_2}, f_{m_3})$, and so on.

Since by 3 this is not a winning strategy for ONE there is a play

$$F(\emptyset), f_{m_2}, F(f_{m_2}), f_{m_3}, F(f_{m_2}, f_{m_3}), f_{m_4}, \ldots$$

lost by ONE. This means that the set $\{f_{m_j} : j = 2, 3, \ldots\}$ is in $\Omega^2_{\underline{0}}$. By the way F was defined, this set also is homogeneous of color j for the original coloring f.

Next we prove that 4 implies 1. To begin, choose continuous functions g_m, $m \in \mathbb{N}$ such that on $[\frac{1}{2\cdot m+2}, \frac{1}{2\cdot m+1}]$ f_m has value 1, and on $[0, \frac{1}{2\cdot m+3}] \cup [\frac{1}{2\cdot m}, 1]$ it has value zero. Further notice that A is in $\Omega^i_{\underline{0}}$ if, and only if, $\{|f| : f \in A\}$ is, for $i \in \{1, 2\}$.

Let $(A_n : n \in \mathbb{N})$ be a sequence of elements of $\Omega^1_{\underline{0}}$. We may assume that each is countable, and enumerate it bijectively, say A_n is enumerated as $(f^n_m : m \in \mathbb{N})$. We may also assume that for each m, n, $f^n_m = |f^n_m|$. Also put $A_0 = (f_n : n \in \mathbb{N})$. Then A_0 is an element of $\Omega^1_{\underline{0}}$. Define

$$A := \{g_n + f^n_m : n, m \in \mathbb{N}\}.$$

Then A is an element of $\Omega^1_{\underline{0}}$. Define a coloring ϕ from $[A]^2$ to $\{1, 2\}$ by

$$phi(\{g_{n_1} + f^{n_1}_{m_1}, g_{n_2} + f^{n_2}_{m_2}\}) = \begin{cases} 1 & \text{if } n_1 = n_2 \\ 2 & \text{otherwise} \end{cases}$$

Apply 4 to find a subset B of A and an $i \in \{1, 2\}$ such that on $[\mathrm{B}]^2$ ϕ has the value i, and B is an element of $\Omega^2_{\underline{0}}$. If i were 1, then for any two elements $g_n + f^n_m$ and $g_j + f^j_k$ of B one would have $n = j$, whence on the interval $[\frac{1}{2\cdot j+2}, \frac{1}{2\cdot j+1}]$ each of the elements of B would have value at least 1, contradicting the fact that X is dense in [0,1], and B is in $\Omega^2_{\underline{0}}$. Thus, i is 2. This in turn implies that for any two elements $g_n + f^n_m$ and $g_j + f^j_k$ of B one has $n \neq j$. But then the set $\{f^n_m : g_n + f^n_m \in B\}$ is an element of $\Omega^2_{\underline{0}}$, and can be obtained by choosing an element from each of infinitely many of the terms of the sequence $(A_n : n \in \mathbb{N})$.

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