

SELECTIVE SCREENABILITY GAME AND COVERING DIMENSION

LILJANA BABINKOSTOVA

ABSTRACT. We introduce an infinite two-person game inspired by the selective version of R. H. Bing's notion of screenability. We show how, for metrizable spaces, this game is related to covering dimension.

1. INTRODUCTION

Let X be a topological space. In [3], R. H. Bing introduced the following notion of *screenability*: For each open cover \mathcal{U} of X there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each n , \mathcal{V}_n is a family of pairwise disjoint open sets; for each n , \mathcal{V}_n refines \mathcal{U} and $\cup_{n < \infty} \mathcal{V}_n$ is an open cover of X . In [1], David F. Addis and John H. Gresham introduced the selective version of screenability: For each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of X there is a sequence $(\mathcal{V}_n : n < \infty)$ such that for each n , \mathcal{V}_n is a family of pairwise disjoint open sets; for each n , \mathcal{V}_n refines \mathcal{U}_n and $\cup_{n < \infty} \mathcal{V}_n$ is an open cover of X . It is evident that selective screenability implies screenability.

Selective screenability is an example of the following selection principle which was introduced in [2]: Let S be a set and let \mathcal{A} and \mathcal{B} be families of collections of subsets of the set S .¹ Then $S_c(\mathcal{A}, \mathcal{B})$

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¹Thus, if \mathcal{U} is a member of \mathcal{A} or of \mathcal{B} , then \mathcal{U} is a collection of subsets of S .

denotes the statement that for each sequence $(\mathcal{U}_n : n < \infty)$ of elements of \mathcal{A} there is a sequence $(\mathcal{V}_n : n < \infty)$ such that

- (1) for each n , \mathcal{V}_n is a family of pairwise disjoint sets;
- (2) for each n , \mathcal{V}_n refines \mathcal{U}_n ; and
- (3) $\cup_{n < \infty} \mathcal{V}_n$ is a member of \mathcal{B} .

With \mathcal{O} denoting the collection of all open covers of topological space X , $S_c(\mathcal{O}, \mathcal{O})$ is selective screenability.

Addis and Gresham noted that countable dimensional metrizable spaces are selectively screenable and asked if the converse is true. Roman Pol, in [7], showed that the answer is no. We will now show that the countable dimensional metric spaces are exactly characterized by a game-theoretic version of selective screenability.

The following game, denoted $G_c(\mathcal{A}, \mathcal{B})$, is naturally associated with $S_c(\mathcal{A}, \mathcal{B})$. Players ONE and TWO play as follows: In the n -th inning, ONE first chooses \mathcal{O}_n , a member of \mathcal{A} , and then TWO responds with \mathcal{T}_n which is pairwise disjoint and refines \mathcal{O}_n . A play $(\mathcal{O}_1, \mathcal{T}_1, \dots, \mathcal{O}_n, \mathcal{T}_n, \dots)$ is won by TWO if $\cup_{n < \infty} \mathcal{T}_n$ is a member of \mathcal{B} ; else, ONE wins. We can consider versions of different lengths of this game. For an ordinal number k , let $G_c^k(\mathcal{A}, \mathcal{B})$ be the game played as follows: In the l -th inning ($l < k$), ONE first chooses \mathcal{O}_l , a member of \mathcal{A} , and then TWO responds with a pairwise disjoint \mathcal{T}_l which refines \mathcal{O}_l . A play

$$\mathcal{O}_0, \mathcal{T}_0, \dots, \mathcal{O}_l, \mathcal{T}_l, \dots l < k$$

is won by TWO if $\cup_{l < k} \mathcal{T}_l$ is a member of \mathcal{B} ; else, ONE wins. Thus, the game $G_c(\mathcal{A}, \mathcal{B})$ is $G_c^\omega(\mathcal{A}, \mathcal{B})$.

2. MAIN RESULTS

From now on we assume that the spaces we work with are metrizable. We will see how selective screenability is related to covering dimension by showing that

- (1) a metrizable space is countable-dimensional if and only if TWO has a winning strategy in the game $G_c^\omega(\mathcal{O}, \mathcal{O})$ (Theorem 2.2);
- (2) for each nonnegative integer n , a metrizable space X is $\leq n$ -dimensional if and only if TWO has a winning strategy in $G_c^{n+1}(\mathcal{O}, \mathcal{O})$ (Theorem 2.4).

We will use the following result:

Lemma 2.1 ([6, Theorem II.XI.21.2]). *Let X be a space and let Y be a subspace of X . Let $(V_i : i \in I)$ be a collection of subsets of Y open in Y . Then there is a collection $(U_i : i \in I)$ of open subsets of X such that for each $i \in I$, we have $V_i = Y \cap U_i$, and for each finite subset F of I , if $\bigcap_{i \in F} V_i = \emptyset$, then $\bigcap_{i \in F} U_i = \emptyset$.*

Theorem 2.2. *Let X be a metric space.*

- (1) *If X is countable dimensional, then TWO has a winning strategy in $\mathbb{G}_c^\omega(\mathcal{O}, \mathcal{O})$.*
- (2) *If TWO has a winning strategy in $\mathbb{G}_c^\omega(\mathcal{O}, \mathcal{O})$, then X is countable dimensional.*

Proof of (1): Let X be countable dimensional, i.e., $X = \bigcup_{n < \infty} X_n$ where each X_n is zero-dimensional. We will define a Markov strategy (for definition, see [4]) σ for player TWO: For an open cover \mathcal{U} of X and $n < \infty$, \mathcal{U} is an open cover of X_n . Since X_n is zero-dimensional, find a pairwise disjoint family \mathcal{V} of subsets of X_n open in X_n such that \mathcal{V} covers X_n and refines \mathcal{U} . By Lemma 2.1, choose a pairwise disjoint family $\sigma(\mathcal{U}, n)$ of open subsets of X refining \mathcal{U} such that each element V of \mathcal{V} is of the form $U \cap X_n$ for some $U \in \sigma(\mathcal{U}, n)$. Now TWO plays as follows: In inning 1, ONE plays \mathcal{U}_1 , and TWO responds with $\sigma(\mathcal{U}_1, 1)$, thus covering X_1 . When ONE has played \mathcal{U}_2 in the second inning, TWO responds with $\sigma(\mathcal{U}_2, 2)$, thus covering X_2 , and so on. And in the n -th inning, when ONE has chosen the cover \mathcal{U}_n of X , TWO responds with $\sigma(\mathcal{U}_n, n)$, covering X_n . This strategy evidently is a winning strategy for TWO.

Proof of (2): Let σ be a winning strategy for TWO. Let \mathcal{B} be a base for the metric space X . For each n , let \mathcal{B}_n be the family $\{B \in \mathcal{B} : \text{diam}(B) < \frac{1}{n}\}$. Consider the plays of the game in which, in each inning, ONE chooses for some n a cover of the form \mathcal{B}_n of X .

Define a family $(C_\tau : \tau \in {}^{<\omega}\mathbb{N})$ of subsets of X as

- (1) $C_\emptyset = \bigcap \{\bigcup \sigma(\mathcal{B}_n) : n < \infty\}$;
- (2) for $\tau = (n_1, \dots, n_k)$, $C_\tau = \bigcap \{\bigcup \sigma(\mathcal{B}_{n_1}, \dots, \mathcal{B}_{n_k}, \mathcal{B}_n) : n < \infty\}$.

We will show that $X = \bigcup \{C_\tau : \tau \in {}^{<\omega}\mathbb{N}\}$. Suppose, to the contrary, that $x \notin \bigcup \{C_\tau : \tau \in {}^{<\omega}\mathbb{N}\}$. Let us choose an n_1 such that $x \notin \sigma(\mathcal{B}_{n_1})$. With n_1, \dots, n_k chosen such that $x \notin \sigma(\mathcal{B}_{n_1}, \dots, \mathcal{B}_{n_k})$,

let us choose an n_{k+1} such that $x \notin \sigma(\mathcal{B}_{n_1}, \dots, \mathcal{B}_{n_{k+1}})$, and so on. Then

$$\mathcal{B}_{n_1}, \sigma(\mathcal{B}_{n_1}), \mathcal{B}_{n_2}, \sigma(\mathcal{B}_{n_1}, \mathcal{B}_{n_2}), \dots$$

is a σ -play lost by TWO, contradicting the fact that σ is a winning strategy for TWO.

Also, we will show that each C_τ is zero-dimensional. Let $x \in C_\tau$ and let $\tau = (n_1, \dots, n_k)$ be given. Thus, x is a member of $\cap\{\cup\sigma(\mathcal{B}_{n_1}, \dots, \mathcal{B}_{n_k}, \mathcal{B}_n) : n < \infty\}$. For each n , choose a neighborhood $V_n(x) \in \sigma(\mathcal{B}_{n_1}, \dots, \mathcal{B}_{n_k}, \mathcal{B}_n)$. Since for each n we have $\text{diam}(V_n(x)) < \frac{1}{n}$, the set $\{V_n(x) \cap C_\tau : n < \infty\}$ is a neighborhood basis for x in C_τ . Also, we have that each $V_n(x)$ is closed in C_τ because of disjointness of TWO's chosen sets. The set $V = \cup\sigma(\mathcal{B}_{n_1}, \dots, \mathcal{B}_{n_k}, \mathcal{B}_n) \setminus V_n(x)$ is open in X and so $C_\tau \setminus V_n(x) = C_\tau \cap V$ is open in C_τ . Thus, each element of C_τ has a basis consisting of clopen sets. \square

Observe that in the proof of Theorem 2.2 we show:

Corollary 2.3. *Let X be a metric space. The following are equivalent.*

- (1) *TWO has a winning strategy in $G_c^\omega(\mathcal{O}, \mathcal{O})$.*
- (2) *TWO has a winning Markov strategy in $G_c^\omega(\mathcal{O}, \mathcal{O})$.*

The proof of the following theorem uses the ideas in the proof of Theorem 2.2.

Theorem 2.4. *Let X be a metric space. The following are equivalent.*

- (1) *If X is $\leq n$ -dimensional then TWO has a winning strategy in $G_c^{n+1}(\mathcal{O}, \mathcal{O})$.*
- (2) *If TWO has a winning strategy in $G_c^{n+1}(\mathcal{O}, \mathcal{O})$, then X is $\leq n$ -dimensional.*

From this theorem, we obtain that the metric space X is n -dimensional if and only if TWO has a winning strategy in $G_c^{n+1}(\mathcal{O}, \mathcal{O})$ but not in $G_c^n(\mathcal{O}, \mathcal{O})$.

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DEPARTMENT OF MATHEMATICS; BOISE STATE UNIVERSITY; BOISE, ID 83725

E-mail address: `liljanab@math.boisestate.edu`