1 Iterabiity

Following Steel [**?**] define a *(coarse) premouse* to be a transitive structure $\langle M, \delta^M\rangle$ satisfying:

- $M \models \mathrm{ZC} + \Sigma_2$-replacement.
- If $F : M||\delta^M \to M \cap \mathrm{OR}$ is M definable from parameters, then F is bounded in $M \cap \mathrm{OR}$. (I will use $M||\gamma$ to mean V^M_γ.)
- $M \models \delta^M$ is inaccessible.

Suppose M and N are premice and $j : M \to N$ a nontrivial elementary embedding. Let $\kappa = \mathrm{crit}(j)$ and $\lambda \le j(\kappa)$. Set

$$E_j = \{(a, x) : a \in [\lambda]^{<\omega} \ \&\ x \in \mathcal{P}([\kappa]^{|a|}) \cap M \ \&\ a \in j(x)\}$$

E_j is the (κ, λ)-M-extender derived from j.

We need a little notation to facilitate the definition of pre-extender. For $a \subseteq b$ finite sets of ordinals say $b = \{\gamma_0, \gamma_2, \ldots, \gamma_{k-1}\}$ with $\gamma_0 < \gamma_2 < \cdots < \gamma_{k-1}$ and $a = \{\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_m}\}$ with $0 \le i_1 < i_2 < \cdots < i_m \le k-1$ define $\pi_{b,a} : [\mathrm{OR}]^{|b|} \to [\mathrm{OR}]^{|a|}$ by

$$\pi_{b,a}(\{\xi_0, \xi_2, \cdots, \xi_{k-1}\}) = \{\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_m}\}$$

where $\xi_0 < \xi_2 < \cdots < \xi_{k-1}$. Clearly, $\pi_{b,a} = \pi_{k,\{i_1,\ldots,i_m\}}$. For $u \in [\mathrm{OR}]^{|b|}$ let $u^{b,a} = \pi_{b,a}(u)$. If $\xi \in a$ set $u^{a,\xi}$ to be that x such that $\{x\} = u^{a,\{\xi\}}$.

We will also use the notation $x^{a,b} = \{u^{a,b} : u \in x\}$ for $x \subseteq [\mathrm{OR}]^{|a|}$, so $x^{a,b} = \pi_{b,a}(x) = \pi^{-1}_{b,a}[x]$. Similarly for $f : [\mathrm{OR}]^{|a|} \to X$ define $f^{a,b} : [\mathrm{OR}]^{|b|} \to X$ by $f^{a,b}(u) = f(u^{b,a})$. For U a measure on $[\kappa]^{|b|}$ set $U^{b,a} = \{x : x^{a,b} \in U\}$. This is the (b, a)-*projection* of U to $[\kappa]^{|a|}$. $U^{b,a}$ makes sense for any $U \subset \mathcal{P}([\kappa]^{|b|})$, but we will only use this notation for measures.

Definition 1. E is a (κ, λ)-pre-extender over M if

- $E \subseteq \bigcup_{n<\omega}[\lambda]^n \times \mathcal{P}([\kappa]^n)$
- For $a \in [\lambda]^{<\omega}$, $E_a = \{x : (a, x) \in E\}$ is a κ-complete ultrafilter on $\mathcal{P}([\kappa]^{|a|}) \cap M$.

- (coherence) For $a \subseteq b \in [\lambda]^{<\omega}$, E_a is the (b,a)-projection of E_b, that is, $E_a = \{x \subseteq [\kappa]^{|a|} : x^{a,b} \in E_b\}$.
- (normality) If $f : [\kappa]^{|a|} \to \kappa$ is such that $\{u : f(u) < \sup(u)\} \in E_a$, then there is $\xi < \sup(a)$ such that $\{u : f^{a\cup\{\xi\},a}(u) = u^{a\cup\{\xi\},\xi}\} \in E_{a\cup\{\xi\}}$.

Given a (κ,λ) M-pre-extender one can form $\operatorname{ult}(M,E)$ with elements of the form $[a,f]^M_E$ for $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to M$ in M in the usual fashion and get $i_E : M \to \operatorname{ult}(M,E)$ defined by $i_E(x) = [a, c_x]^M_E$ where c_x is the constant x function. Assuming $\operatorname{wfp}(\operatorname{ult}(M,E))$ has been collapsed to a transitive structure, normality gives that $a = [a, \operatorname{id}\restriction[\kappa]^{|a|}]^M_E$ so that $\lambda \subseteq \operatorname{wfp}(\operatorname{ult}(M,E))$ and $[a,f]^M_E = i_E(f)(a)$. It is clear that $\kappa = \operatorname{crit}(i_E)$.

If $\operatorname{ult}(M,E)$ is wellfounded, then E is called a (κ,λ) *M-extender*. I will write $M \xrightarrow{i}_E N$ to denote E is an M-extender, $N = \operatorname{ult}(M,E)$, and i is the ultrapower embedding.

A (κ,λ) M-pre-extender is *weakly amenable* if for all $f : \kappa \to \mathcal{P}([\kappa]^n)$ in M, $\{\gamma < \kappa : f(\gamma) \in E_a\} \in M$. This is equivalent to

$$M||\kappa+1 = \operatorname{ult}(M,E)||\kappa+1$$

Let $j : M \to N$ with N transitive, $\operatorname{crit}(j) = \kappa$, $\lambda \leq j(\kappa)$, and $E = E_j$ be the (κ,λ)-derived extender. Let $k : \operatorname{ult}(M,E) \to N$ be the factor embedding given by $k(i_E(f)(a)) = j(f)(a)$. Then $\gamma = \operatorname{crit}(k) \geq \lambda$ and

$$H(\gamma)^N = H(\gamma)^{\operatorname{ult}(M,E)}$$

For E a (κ,λ) M-extender $\operatorname{crit}(E) = \kappa$, $\operatorname{lh}(E) = \lambda$ is the *length of* E, and $\operatorname{str}^M(E) = \sup\{\gamma : M||\gamma \subseteq \operatorname{ult}(M,E)\}$ is the strength of E. If $E \in M$, then $\operatorname{str}^M(E) \leq \operatorname{lh}(E) \leq i^M_E(\kappa)$ since $E \subseteq M||\lambda$ and $E \notin \operatorname{ult}(M,E)$.

Lemma 2. Suppose M and N are premice and E is a (κ,λ) M-extender. If $M||\kappa+1 = N||\kappa+1$, then $\operatorname{ult}(N,E)$ makes sense. We have the following agreement:

- $i^M_E \restriction \kappa+1 = i^N_E \restriction \kappa+1$ and $\operatorname{ult}(M,E)||i^M_E(\kappa)+1 = \operatorname{ult}(N,E)||i^N_E(\kappa)+1$.
- $\operatorname{ult}(N,E)||\operatorname{str}^M(E) = M||\operatorname{str}^M(E)$.
- $\operatorname{ult}(N,E)||\operatorname{str}^M(E)+1 \neq M||\operatorname{str}^M(E)+1$. $\dashv$

Definition 3. A *tree order* on θ is an order $<_T$ such that:

- $\alpha <_T \beta \implies \alpha < \beta$.
- α is a T successor $\iff$ α is a successor ordinal.
- $\beta < \theta$ a limit ordinal $\implies$ $\{\alpha : \alpha <_T \beta\}$ is cofinal in β.

θ is called the *length of* T and is denoted $\mathrm{lh}(T)$. The T predecessor of $\alpha + 1$ will be denoted α_*. $\dashv$

Definition 4. $\mathcal{T}$ is a *iteration tree* on a premouse M if $\mathcal{T} = \langle T, \langle E_\alpha : \alpha + 1 < \mathrm{lh}(T)\rangle\rangle$ where

- $M_0 = M$,
- T is a tree order,
- $E_\alpha \in M_\alpha || \delta^{M_\alpha}$ is an $(\kappa_\alpha, \lambda_\alpha)$ M_α-extender, and
- $M_{\alpha^*} || \kappa_\alpha + 1 = M_\alpha || \kappa_\alpha + 1$ and $M_{\alpha+1} = \mathrm{ult}(M_{\alpha^*}, E_\alpha)$.

Here we are assuming all of the models appearing are wellfounded. If required to avoid ambiguity, a superscript $\mathcal{T}$ can be included, e.g., $M^{\mathcal{T}}_{\alpha+1} = \mathrm{ult}(M^{\mathcal{T}}_{\alpha^{\mathcal{T}}_*}, E^{\mathcal{T}}_\alpha)$. $\dashv$

Let $\rho_\alpha = \mathrm{str}^{M_\alpha}(E_\alpha)$ and for $\lambda \leq \theta = \mathrm{lh}(\mathcal{T})$ set

$$\rho[\alpha, \lambda) = \inf\{\rho_\gamma : \alpha \leq \gamma < \lambda\}$$

Lemma 5. If $\mathcal{T}$ is an iteration tree on M, then for $\alpha < \beta < \mathrm{lh}(\mathcal{T})$

- $M_\alpha || \rho[\alpha, \beta) = M_\beta || \rho[\alpha, \beta)$
- $M_\beta || \rho[\alpha, \beta) + 1 \subsetneq M_\alpha || \rho[\alpha, \beta) + 1$

Proof. This is proved by induction on β. If $\beta = \gamma + 1$, then $M_{\gamma+1} || \rho_\gamma = M_\gamma || \rho_\gamma$ and $M_{\gamma+1} || \rho_\gamma + 1 \subsetneq M_\gamma || \rho_\gamma + 1$ by lemma 2. So the claim follows by induction.

If β is a limit we will prove a bit more.

Claim 1. Let b be any cofinal branch in $T|\beta$ such that $\rho[\alpha, \beta) \in \mathrm{wfp}(M_b)$, then $M_b || \rho[\alpha, \beta) = M_\alpha || \rho[\alpha, \beta)$.

Notice that for $\xi \in b$

$$\mathrm{crit}(i_{\xi,b}) = \inf\{\kappa_\gamma : \gamma + 1 \in b \setminus \xi\}$$

so

$$\sup_{\xi\in b} \mathrm{crit}(i_{\xi,b}) = \sup_{\xi\in b} \inf\{\kappa_\gamma : \gamma + 1 \in b \setminus \xi\} = \mu_b$$

We actually see that $\rho[\alpha,\beta) \leq \mu_b$. Suppose $\mu_b < \rho[\alpha,\beta)$, then for cofinally many γ in b, $\mathrm{crit}(i_{\gamma_*}, b) = \kappa_\gamma < \rho[\alpha,\beta) \leq \rho_\gamma$, so $\rho[\alpha,\beta) \notin \mathrm{rng}(i_{\gamma_*,b})$. But this is nonsense as there must be $\xi \in b$ and $\bar{\rho}$ so that $i_{\xi,b}(\bar{\rho}) = \rho[\alpha,\beta)$. Choose $\xi \in b \setminus \alpha + 1$ with $\mathrm{crit}(i_{\xi,b}) \geq \rho[\alpha,\beta)$, then $M_\xi || \rho[\alpha,\beta) = M_b || \rho[\alpha,\beta)$ and $M_\xi || \rho[\alpha,\beta) = M_\alpha || \rho[\alpha,\beta)$ so we have the desired agreement. ❑

Definition 6. For $\mathcal{T}$ an iteration tree of limit length θ define

$$\delta(\mathcal{T}) = \sup\{\rho[\alpha,\theta) : \alpha < \theta\}$$

and the *common part of* $\mathcal{T}$ is defined to be

$$M(\mathcal{T}) = \bigcup_{\alpha<\theta} M_\alpha || \rho[\alpha,\theta)$$

The above proof actually proves the following lemma.

Lemma 7. The above argument showed that if b is any cofinal branch of $\mathcal{T}$ and $\delta(\mathcal{T}) \subseteq \mathrm{wfp}(M_b)$, then

$$M_b || \delta(\mathcal{T}) = M(\mathcal{T})$$

and

$$\sup_{\xi\in b} \mathrm{crit}(i_{\xi,b}) = \sup_{\xi\in b} \inf\{\kappa_\gamma : \gamma + 1 \in b \setminus \xi\} \geq \delta(\mathcal{T})$$

something stronger holds, namely,

$$\sup_{\xi\in b} \mathrm{crit}(i_{\xi,b}) = \sup_{\xi\in b} \inf\{\kappa_\gamma : \gamma + 1 \in b \setminus \xi\} = \delta(\mathcal{T})$$

Proof. To see this let $\xi \in b$. We want to see that $\mathrm{crit}(i_{\xi,b}) < \delta(\mathcal{T})$. Take $\gamma > \xi$ so that $\rho_\gamma \leq \delta(\mathcal{T})$. If $\gamma + 1 \in b$, then $\rho_\gamma > \kappa_\gamma \geq \mathrm{crit}(i_{\xi,b})$ as desired. Otherwise take $\beta + 1 \in b \setminus \gamma + 1$ minimal. We have $\beta_* \leq \gamma$. We have $\kappa_\beta < \rho[\beta_*, \beta)$ and $\beta > \gamma$ so $\mathrm{crit}(i_{\xi,\beta}) \leq \mathrm{crit}(i_{\beta_*,b}) < \delta(\mathcal{T})$. ❑

In particular, if b and c are two cofinal branches with $\delta(\mathcal{T}) \subseteq \mathrm{wfp}(M_b) \cap \mathrm{wfp}(M_c)$, then M_b and M_c agree below $\delta(\mathcal{T})$.

Since $\kappa_\alpha < \rho_\gamma$ for all $\gamma \in [\alpha_*, \alpha)$ we have

$$\sup\{\kappa_\alpha : \alpha_* \leq \gamma < \alpha\} \leq \rho_\gamma$$

The tree is called $+n$ if

$$\sup\{\kappa_\alpha : \alpha_* \leq \gamma < \alpha\} + n \leq \rho_\gamma$$

The main result on iterability from [**?**] concerns +2 trees. Any length ω-tree is automatically a +1 tree.

Definition 8. An iteration tree $\mathcal{T}$ on premouse M is *normal* if:

(1) There are ordinals ν_α for $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ such that $\nu_\alpha + 2 \leq \rho_\alpha$ and such that $\alpha < \beta \implies \nu_\alpha < \nu_\beta$. Moreover it is required that $\kappa_\alpha < \nu_{\alpha_*}$.

(2) α_* is least γ such that $\kappa_\alpha < \nu_\gamma$. (This is sometimes separated from normality and called maximality, but maximal will be used later for a different notion and hence I will take this as part of normality. In general an iteration tree is maximal if α_* is taken to be as small as possible, perhaps preserving some property like being "+n" or "normal".) $\dashv$

Normal trees are +2-trees since $\kappa_\alpha < \nu_\gamma$ for all $\gamma \in [\alpha_*, \alpha)$ so $\sup\{\kappa_\alpha : \alpha_* \leq \gamma < \alpha\} \leq \nu_\gamma$, hence $\sup\{\kappa_\alpha : \alpha_* \leq \gamma < \alpha\} + 2 \leq \nu_\gamma + 2 \leq \rho_\gamma$. A transfinite stack of +2-trees is itself a +2 so a transfinite stacks of normal trees is a +2 tree. Notice that if $\alpha + 1 <_T \beta + 1$ and T is normal, then $\kappa_\beta \geq \nu_\alpha$ so the critical points are increasing.

Definition 9. A premouse M is called *normally ξ-iterable* if player II wins the normal iteration game $G(M, \xi)$ defined as follows:

- At successor stages $\alpha + 1 < \xi$ player I either plays "end" and the game ends with a win for II or plays E_α from the current model M_α satisfying $\nu_\alpha > \nu_\beta$ for all $\beta < \alpha$. Letting α_* be the least γ so that $\kappa_\alpha < \nu_\gamma$, then if $\mathrm{ult}(M_{\alpha_*}, E_\alpha)$ is illfounded I wins.
- At limit stages $\lambda < \xi$, II must play a cofinal wellfounded branch b and $M_\lambda = M_b$.

If the game lasts θ-moves, then II wins. $\dashv$

The plays of $G(M,\xi)$ are normal iteration trees of length $\leq \xi$. We also want a game that determines stacks of normal trees.

Definition 10. A premouse M is called *normally* (ξ,ζ)*-iterable* if player II wins the normal iteration game $G(M,\xi,\zeta)$ defined as follows:

At the beginning round $\alpha < \zeta$ there is a current last model M_α. Here there are two cases, either α is a limit and M_α is the direct limit of $\{\langle M_\beta : \beta < \alpha\rangle, \langle i_{\gamma,\beta} : \gamma < \beta < \alpha\rangle\}$ coming from the preceding rounds. In this case if M_α is not wellfounded, then II looses. Else, there is a last tree $\mathcal{T}_\alpha$ on M_α of length $\theta_\alpha + 1 < \xi$ so that $M_\alpha = M^{\mathcal{T}_\alpha}_{\theta_\alpha}$. Now a round of $G(M_\alpha,\xi)$ is played in which I looses if he does not play "exit" at some move before ξ, this determines the beginning of a new round. If I plays "end", then the game ends and II wins.

If the game goes all ζ-rounds and ζ is a limit ordinal, then for II to win, the resulting direct limit must be wellfounded. ⊣

A play of $G(M,\xi,\zeta)$ results in a stack of $\leq \zeta$ normal maximal trees all of which have successor length $< \xi$ except possibly the last one.

Definition 11. For $\mathcal{T}$ an iteration tree on M call b a maximal branch of $\mathcal{T}$ if b is downward closed under $<_T$ and there is no $\beta < \mathrm{lh}(\mathcal{T})$ such that $b \subseteq [0,\beta]_T$. ⊣

Here is the main result on iterability from [**?**].

Theorem 12. Suppose $\langle V_\alpha, \delta\rangle$ is a premouse and $\pi : M \to V_\alpha$ is elementary with $\delta \in \mathrm{rng}(\pi)$ so that if $\delta^M = \pi^{-1}(\delta)$, then $\langle M, \delta^M\rangle$ is a premouse. Suppose that $\mathcal{T}$ is a countable $+2$ tree on M, then either:

- $\mathcal{T}$ has a maximal branch b such that there is $k : M_b \to V_\alpha$ such that $k \circ i_b = \pi \restriction M$. (Such a branch is called π*-realizable.*)
- There is no maximal π-realizable branch and $\mathrm{lh}(\mathcal{T}) = \theta + 1$ and there is $k : M_\theta \to V_\alpha$ so that $k \circ i_{0,\theta} = \pi \restriction M$. ($M_\theta$ is called π-realizable.) ⊣

From now on "tree" will mean $+2$-tree. In some cases this theorem gives an iteration strategy for M.

Definition 13. (1) κ is λ*-*A*-strong* if there is $k : V \to M$ such that $V_\lambda \subseteq M$ and $k(A) \cap \lambda = A \cap \lambda$. Such an embedding is coded in a (κ,γ)-extender for $\gamma \geq |V_\lambda|$.

(2) κ is $<\delta$*-*A*-strong* if $\{\lambda < \delta : \kappa \text{ is } \lambda\text{-}A\text{-strong}\}$ is cofinal in δ.

(3) δ *is Woodin* if for all $A \subseteq \delta$, there is a $\kappa < \delta$ that is $<\delta$-A-strong. $\dashv$

If δ is Woodin, then the sets $S_A = \{\kappa < \delta : \kappa \text{ is } {<}\delta\text{-}A\text{-strong}\}$ generate a normal, hence $<\delta$-complete, filter $\mathcal{F}_\delta$ on δ and so clearly δ is regular. Clearly, $\mathcal{F}_\delta$ concentrates on the measurable cardinals $< \delta$. It is also clear that for any $\gamma < \delta$, A can be chosen so that $\min(S_A) > \gamma$ so that δ is a limit of measurables and hence is inaccessible. To see this just let $A = \delta \setminus (\gamma + 1)$. Let $\lambda > \gamma$ and $j : V \to M$ witness κ is λ-A-strong. Then $j(A) \cap \lambda = A \cap \lambda = \lambda \setminus (\gamma + 1)$. If $\kappa < \gamma$, then $j(A) = j(\delta) \setminus (j(\gamma) + 1)$, but $j(\gamma) > \lambda$ so $j(A) \cap \lambda = \emptyset$.

For δ Woodin, since δ is inaccessible the extenders witnessing strength can be taken from V_δ. Thus being Woodin is Π^1_1 over V_δ and hence Woodin cardinals need not be weakly compact.

For $\mathcal{A} \subseteq \mathcal{P}(\delta)$ say δ is Woodin with respect to sets in $\mathcal{A}$ if $S_A \neq \emptyset$ for all $A \in \mathcal{A}$.

THEOREM 14. Suppose b and c are cofinal branches of limit length normal tree $\mathcal{T}$. Suppose $\delta(\mathcal{T}) \in \text{wfp}(M_b) \cap \text{wfp}(M_c)$, and $A \subseteq \delta$ is in both M_b and M_c. Then $\langle M(\mathcal{T}), A\rangle \models S_A \neq \emptyset$. So $\delta(\mathcal{T})$ is Woodin with respect to $\mathcal{P}(\delta(\mathcal{T})) \cap M_b \cap M_c$. (This theorem holds for arbitrary iteration trees with a slightly more involved proof using lemma 7 in place of normality.) $\dashv$

Proof. Let $\alpha \geq \sup(b \cap c)$ be large enough so that for any $\beta \in b$ and $\gamma \in c$, $A \in \text{rng}(i_{\beta,b}) \cap \text{rng}(i_{\gamma,c})$. For $\beta + 1 \in b$ and $\gamma + 1 \in c$ with $\beta_*, \gamma_* \geq \alpha$ and $\kappa = \min\{\text{crit}(i_{\beta_*,b}), \text{crit}(i_{\gamma_*,c})\}$

CLAIM 2.

$$\text{rng}(i_{\beta_*,b}) \cap \text{rng}(i_{\gamma_*,c}) \cap \delta(\mathcal{T}) = \kappa$$

This is called the ”zipper” argument. Without loss of generality assume $\kappa = \text{crit}(i_{\beta_*,b}) = \text{crit}(E_\beta)$. Let $\beta_0 = \beta$. Let γ_0 be minimal such that $\gamma_0 + 1 \in c \setminus \beta_0 + 1$ and notice $\gamma_{0*} \leq \beta_0$. This tells us that $\kappa_{\gamma_0} < \nu_{\beta_0}$. Let β_1 be least such that $\beta_1 + 1$ is in $b \setminus \gamma_0 + 1$. Then $\beta_{1*} \leq \gamma_0$ and so $\kappa_{\beta_1} < \nu_{\gamma_0}$. continue in this fashion

$$\gamma_i \text{ is least such that } \gamma_i + 1 \in c \setminus \beta_i + 1$$
$$\beta_i \text{ is least such that } \beta_i + 1 \in b \setminus \gamma_{i-1} + 1$$

So $\beta_0 < \gamma_0 < \beta_1 < \gamma_1 < \cdots$. This will give

$$\kappa_{\gamma_i} < \nu_{\beta_i} < \nu_{\gamma_i} \quad \text{and } \kappa_{\beta_{i+1}} < \nu_{\gamma_i} < \nu_{\beta_{i+1}}$$

since b and c are closed it is clear that the β_i's and γ_i's are cofinal in b and c respectively so that

$$\delta(\mathcal{T}) = \sup\{\nu_{\gamma_i} : i \in \omega\} = \sup\{\nu_{\beta_i} : i \in \omega\} = \sup\{\kappa_{\gamma_i} : i \in \omega\} = \sup\{\kappa_{\beta_i} : i \in \omega\}$$

So $[\kappa, \delta(\mathcal{T}) = \bigcup_i [\kappa_{\beta_i}, \nu_{\beta_i}) \cup [\kappa_{\gamma_i}, \nu_{\gamma_i})$. On the other hand $[\kappa_i, \nu_{\beta_i}) \cap \operatorname{rng}(i_{\beta_0,b}) = \emptyset$ and $[\kappa_{\gamma_i}, \nu_{\gamma_i}) \cap \operatorname{rng}(i_{\gamma_0,c}) = \emptyset$ so this proves the claim and hence the "zipper" part of this argument.

❑

Call a premouse M *suitable* if

- $M = L(M||\delta^M)$, and
- for all $\alpha < \delta^M$, $L(M||\alpha) \models$ "α is not Woodin".

Call a limit length tree $\mathcal{T}$ on a suitable M *maximal* if $L(M(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin. Call $\mathcal{T}$ *short* if for all limit $\lambda < \operatorname{lh}(\mathcal{T})$, $L(M(\mathcal{T} \upharpoonright \lambda)) \models$ "$\delta(\mathcal{T} \upharpoonright \lambda)$ is not Woodin".

THEOREM 15. For M suitable and $\mathcal{T}$ a limit length short tree on M there is at most one cofinal wellfounded branch, which we will denote $b_{\mathcal{T}}$. Moreover, either

(1) $b_{\mathcal{T}}$ exists and $b_{\mathcal{T}} \in L(M, \mathcal{T})$, or

(2) $\mathcal{T}$ is maximal.

Moreover, in case (2) if there is a cofinal wellfounded branch anywhere, then there is one in $L(\langle M, \mathcal{T}\rangle^\sharp)$ (if $\langle M, \mathcal{T}\rangle^\sharp$ exists), and in addition, $M_{b_{\mathcal{T}}} = L(M(\mathcal{T}))$ so that in any case the model $M_{b_{\mathcal{T}}}$ is trivial to compute, even if $b_{\mathcal{T}}$ is not. ⊣

Remark. If M is suitable, then there is a tree in M of length δ^M with no branch in M. Moreover, if M is κ^+-iterable, then sharps for subsets of κ exist in V. Both of these facts come from Woodin's generic iteration argument.

I will prove this theorem in the course of what follows.

LEMMA 16. Let M be suitable and $\mathcal{T}$ a limit length short tree on M. For every α there is a generic cofinal α-wellfounded branch b, i.e., $\alpha \in \operatorname{wfp}(M_b)$.

Proof. Suppose there is an α-bad short tree $\mathcal{T}$ on M in V. Take $\theta \geq (\max\{\mathrm{lh}(\mathcal{T}), |M||\delta^M|, \alpha\})^+$ regular. Fix $\zeta > \theta$ regular. Let $g \subseteq \mathrm{Col}(\omega, \zeta)$ be V-generic. In $V[g]$ there is N with $\zeta \in \mathrm{wfp}(N)$ and such that in N there is a short normal tree $\mathcal{T}$ of length $< \theta$ on $L_\zeta(M||\delta^M)$ with $N||\zeta^{\mathrm{Col}(\omega,\theta)}$ having no α-wellfounded branches. The existence of such an N is a Σ^1_1 fact about a real coding ζ and $M||\delta^M$, thus $M^{\mathrm{Col}(\omega,\zeta)}$ will see N with $\zeta \in \mathrm{wfp}(N)$ with $\mathcal{T}$ of length $< \theta$ on $L_\zeta(M||\delta^M)$ such that in $N||\zeta^{\mathrm{Col}(\omega,\theta)}$ there are no α-wellfounded branches. Let $\xi > \zeta$ be such that $M||\xi$ is sufficiently elementary in M to capture the above.

Let $X \prec M||\xi$ with $M||\delta^M, \alpha, \theta, \zeta \in X$ and let $\pi : \bar{M} \to M||\xi$. Let $\bar{g} \subseteq \mathrm{Col}(\omega, \bar{\zeta})$ be $\bar{M}$-generic with $\bar{g} \in M$. In $\bar{M}[\bar{g}]$ take N with $\bar{\zeta} \in \mathrm{wfp}(N)$ and $\mathcal{T} \in N$ a tree of length $< \bar{\theta}$ such that $N||\bar{\zeta}^{\mathrm{Col}(\omega,\bar{\theta})}$ has no $\bar{\alpha}$-wellfounded branch.

Claim. $\mathcal{T}$ lifts to a tree on $\bar{M}$.

Suppose for a moment that this claim is true. We know that there is a maximal branch b in M which is π-realizable. This branch must be cofinal, from essentially the argument below that shows that $\mathcal{T}$ lifts. But then there is a $\bar{\alpha}$-wellfounded branch through $\mathcal{T}$ and this is a Σ^1_1 fact about a real coding $\bar{M}||\theta$, $\mathrm{lh}(\mathcal{T})$, and $\bar{\alpha}$ and hence true in $N||\bar{\zeta}^{\mathrm{Col}(\omega,\bar{\theta})}$. This is a contradiction.

To see that $\mathcal{T}$ lifts suppose otherwise and let $\gamma < \mathrm{lh}(\mathcal{T})$ be least such that $\mathcal{T}$ fails to lift. We only need to see that $\bar{M}_\gamma$ is wellfounded. The point is that there can be no limit $\lambda < \mathrm{lh}(\mathcal{T})$ such that $\mathcal{T} \restriction \lambda$ has two wellfounded cofinal branches since this would give $N||\bar{\zeta}$ would not think $\mathcal{T}$ is short. ❑

Parts (1) and (2) of Theorem 15 now follows. Either for all large enough α the α-wellfounded cofinal branches agree, hence there is one wellfounded branch, or we can find distinct α-wellfounded cofinal branches for $\alpha > \delta(\mathcal{T})^+$ and hence $L_\alpha(M(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is Woodin ", but $\mathcal{P}(\delta(\mathcal{T})) \cap L(M(\mathcal{T})) = \mathcal{P}(\delta(\mathcal{T})) \cap L_\alpha(M(\mathcal{T}))$. In the first case the branch $b_\mathcal{T}$ is in $L(M, \mathcal{T})$, in the second case $\mathcal{T}$ is maximal. The remainder of the theorem will be dealt with in the following section.

1.1 Computing the true branch through normal maximal trees.

Fix a premouse and a maximal short tree $\mathcal{T}$ on M that has a cofinal wellfounded branch $b_\mathcal{T}$. We want to see how to compute $b_\mathcal{T}$.

For $s \in [\mathrm{OR}]^{<\omega}$ define

$$\gamma_s^M = \sup\{\xi < \delta^M : \xi \text{ is definable in } M||\alpha_s \text{ from } s \cup \{\delta^M\}\}$$

where α_s is the least ordinal greater than $\sup(s)$ such that $M||\alpha_s$ satisfies ZFC* (some reasonable fragment of ZFC.) Define

$$X_s^M = \mathrm{Hull}^{M||\alpha_s}(\gamma_s^M \cup s \cup \{\delta^M\})$$

Notice

$$X_s^M \cap M||\delta^M = M||\gamma_s^M$$

Since δ^M is inaccessible it suffices to see that $X_s^M \cap \delta^M = \gamma_s^M$. Suppose $\zeta \in X_s^M \cap \delta^M \setminus \gamma_s^M$, then ζ is Σ_i-definable from $\bar{\beta} \in [\gamma_s^M \cup s \cup \{\delta^M\}]^{<\omega}$. We can find $\gamma < \gamma_s^M$ definable from $s \cup \cup \{\delta^M\}$ and then define ζ_γ to be the supremum of all those things Σ_i-definable by parameters in $\gamma \cup s \cup \{\delta^M\}$. Then ζ_γ is definable from $s \cup \{\delta^M\}$ and $\zeta < \zeta_\gamma$ so $\zeta < \gamma_s^M$.

LEMMA 17. Suppose $\mathcal{T}$ is a short maximal tree on M. Let $s \in [\mathrm{OR}]^{<\omega}$. Let b and c be generic $\sup(s) + \omega$-wellfounded branches which fix s and sends δ^M to $\delta((T))$. Then $i_b \restriction X_s^M = i_c \restriction X_s^M$.

Proof. Let $b \cap c = [0, \gamma]_{\mathcal{T}}$. Let $\xi + 1 \in b$ and $\zeta + 1 \in c$ with $\zeta_* = \xi_* = \gamma$. Let $\kappa = \min\{\kappa_\zeta, \kappa_\xi\}$. The "zipper" argument shows that $\mathrm{rng}(i_b) \cap \mathrm{rng}(i_c) \cap \delta(\mathcal{T}) = \kappa$. This implies that $\gamma_s^{M_\gamma} \leq \kappa$ so $i_b \restriction (M||\gamma_s^M) = i_c \restriction (M||\gamma_s^M)$ which proves the lemma. ❑

DEFINITION 18. Call s $\mathcal{T}$-good if there is a $\sup(s) + \omega$-wellfounded generic cofinal branch sending δ^M to $\delta(\mathcal{T})$. For $\Gamma \subseteq \mathrm{OR}$ call Γ $\mathcal{T}$-good if for all $s, t \in \Gamma^{<\omega}$, $s \cup t$ is $\mathcal{T}$-good. ⊣

Suppose $\mathcal{T}$ is a short maximal tree on a suitable M. Recall that $M_\infty = L(M(\mathcal{T}))$ must be the branch model for any cofinal wellfounded branch if such exists. If such a branch b exists, then it must be the case that

- There is a proper class Γ_b of fixed points of b.
- $i_b(\delta^M) = \delta(\mathcal{T})$.

An easy absoluteness argument gives that for all $s \in \Gamma_b$ there is a generic $\sup(s) + \omega$-wellfounded cofinal branch fixing s and sending δ^M to $\delta(\mathcal{T})$. thus $L(M, \mathcal{T})$ sees that s is $\mathcal{T}$-good. Of course there is no reason $L(M, \mathcal{T})$ should see Γ.

Assuming s is $\mathcal{T}$-good, we have $i^s : X_s^M \to X_s^{M_\infty}$ which must agree with any branch embedding which fixes s and sends δ^M to $\delta(\mathcal{T})$, again if any exists at all.

If Γ is $\mathcal{T}$-good, then there is a natural system associated to Γ. For s in $\Gamma^{<\omega}$ we have $i^{s,t} : X_s^M \to_{\Sigma_0} X_t^{M_\infty}$. Let $X_\Gamma^M = \bigcup_{s\in\Gamma^{<\omega}} X_s^M$ and similarly define $X_\Gamma^{M_\infty}$. We have $i : X_\Gamma^M \to X_\Gamma^{M_\infty}$. Notice $X_\Gamma^M \cap (M||\delta^M)$ is transitive. Let $\gamma_\Gamma^M = \sup_{s\in\Gamma^{<\omega}} \gamma_s^M$, then $X_\Gamma^M||\delta^M = M||\gamma_\Gamma^M$. If the collapse of X_Γ^M is L-closed (or at least $\mathcal{P}(\gamma_\Gamma^M) \cap L(M||\gamma_G^M)$ is contained in the collapse of X_Γ^M), then $\gamma_\Gamma^M = \delta^M$ by suitability of M. Similar comments apply to M_∞.

Definition 19. If $\Gamma \subseteq \mathrm{OR}$ is such that $\mathcal{T}$ is Γ-good and the collapse of both X_Γ^M and $X_\Gamma^{M_\infty}$ are L-closed, then call Γ $\mathcal{T}$-*full.* Call Γ $\mathcal{T}$-*thick* if in addition, $M = X_\Gamma^M$.

Examples of $\mathcal{T}$-thick sets include:

- Any $\mathcal{T}$-good proper class.
- Any infinite set of indiscernibles for $L(M, \mathcal{T})$.

Theorem 20. Suppose M is suitable and that $\mathcal{T}$ is a short and maximal tree on M. The following are equivalent:

1. There is a wellfounded cofinal (necessarily unique) branch $b_\mathcal{T}$.
2. There is a $\mathcal{T}$-good and $\mathcal{T}$-thick set Γ.

Moreover, $b_\mathcal{T} \in L(M, \Gamma)$.

Proof. That (1) implies (2) was explained above, namely, take Γ to be the fixed points of $b_\mathcal{T}$. So assume Γ is an arbitrary $\mathcal{T}$-good and $\mathcal{T}$-full set. It is clear that Γ can be replaced by a set if it is a class, so just assume Γ is a set. Let θ be large (larger than Γ and everything else). In $L(M, \mathcal{T})^{\mathrm{Col}(\omega,\theta)}$ we can find for each $s \in \Gamma^{<\omega}$ a $\sup(s) + \omega$-wellfounded cofinal generic branch b_s. We have made Γ countable so let $s_0 \subset s_1 \cdots$ be sequences from Γ so that every $\gamma \in \Gamma$ appears in some s_i.

Suppose there is a single branch b that is simultaneously s_i-good for all i. Letting $\bar{M} = L_\zeta(M||\delta^M)$ and $\bar{M}_\infty = L_\xi(M(\mathcal{T}))$ be the collapses of X_Γ^M and $X_\Gamma^{M_\infty}$ respectively we have $i_\Gamma : \bar{M} \to \bar{M}_\infty$ which is according to b. In this case the branch will just lift.

We may assume that the ordinals $\xi_i = \sup \bigcup_{k\geq i} b_k$ are increasing. Let c be the "common part" branch, i.e., $c = \bigcup_{i\in\omega}[0, \xi_i]_T$. The goal is to show that c is a cofinal wellfounded branch. Let $\kappa_i = \inf\{\mathrm{crit}(i_{\xi_i,b_{s_i}}), \mathrm{crit}(i_{\xi_i,b_{s_{i+1}}})\}$, then $\gamma_i = \gamma_{s_i}^{M_{\xi_i}} \leq \kappa_i$ so $\gamma_i = \gamma_{s_i}^{M_\infty} = \gamma_{s_i}^{M_c}$ and $X_{s_i}^{M_c}||\gamma_i = X_{s_i}^{M_\infty}||\gamma_i = X_{s_i}^{M_{\xi_i}}||\gamma_i$. Since $\sup_i \gamma_i = \delta(\mathcal{T})$, c is cofinal. We can now embed M_c into $X_\Gamma^{M_\infty}$ so c is a cofinal wellfounded branch. ❑

2 Quasi Iterations