TOPOLOGICAL GROUPS AND COVERING DIMENSION

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ABSTRACT. We consider a natural way of extending the Lebesgue covering dimension to various classes of infinite dimensional topological groups. The dimension function that we introduce extends Lebesgue covering dimension, has the hereditary property, and has a product theory that is more similar to the product theory for the finite dimensional case.

All spaces in this paper are assumed to be separable and metrizable. In a previous paper [5] we introduced a dimension function called *game dimension* and denoted $\mathsf{dim}_{\mathsf{G}}(\cdot)$. Game dimension is an extension of Lebesgue covering dimension, denoted dim. For a separable metrizable space X, $\mathsf{dim}_{\mathsf{G}}(X) \leq \omega_1$ (Theorem 21 of [5]), and when $\mathsf{dim}_{\mathsf{G}}(X) < \omega_1$, then in fact X is a selectively screenable space[1] (Theorem 22 of [5]). Also, an infinite dimensional separable metric space X is countable dimensional if, and only if, $n = \mathsf{dim}_{\mathsf{G}}(X) = \omega$ (Theorem 2.2 of [2]). Though the theory of game dimension has some nice properties it has two drawbacks: Unlike in the finite or countable case, game dimension does not have nice hereditary properties for spaces with game dimension larger than ω, and its product theory also contains properties not present for the finite dimensional case it generalizes.

The purpose of this paper is to introduce a dimension function which extends Lebesgue covering dimension, has the hereditary property, and has a product theory that is more similar to the product theory for the finite dimensional case. The classical embedding theorem of Menger and Nöbeling provides one inspiration for this new dimension function, to be denoted $\mathsf{dim}_{\mathsf{nbd}}(\cdot)$: Every separable metric space X of covering dimension n is homeomorphic to a subspace of $\mathbb{R}^{2\cdot n+1}$. Now $\mathbb{R}^{2\cdot n+1}$ is a topological group under addition. A game theoretic characterization of finite dimension n in topological groups ([4], Theorem 15) is the second source of inspiration for our new dimension function. Relative to the group $\mathbb{R}^{2\cdot n+1}$ we obtain $\mathsf{dim}(X) = \mathsf{dim}_{\mathsf{nbd}}(X)$. Indeed, for any topological group $(H, *)$ such that finite dimensional metric space X is homeomorphic to a subspace of H we have $\mathsf{dim}(X) = \mathsf{dim}_{\mathsf{nbd}}(X)$.

Given a topological group $(H, *)$ we define an infinite game on $(H, *)$ and use the length required so that TWO would have a winning strategy to define a dimension for H. Corresponding dimension values can be defined for topological subspaces X of H. When the group $(H, *)$ is clear from context, the dimension of X as derived from the game is denoted $\mathsf{dim}_{\mathsf{nbd}}(X)$.

For the class of separable metric spaces there are several choices of a separable metric group $(H, *)$ such that each separable metric space is homeomorphic to a subspace of the group. The group $(\mathbb{R}^{\mathbb{N}}, +)$ is such an example. By the classical theorem of Banach and Mazur that every separable metric space embeds isometrically into the topological group $(\mathsf{C}[0,1], +)$ of continuous real-valued functions on

[1] We define the notion in the next section

the unit interval endowed with the supremum norm, is another example. Any such group is a candidate relative to which a corresponding dimension function $\mathsf{dim}_{\mathsf{nbd}}$ may be defined. For the results presented in this paper the reader may use either of these groups as the group relative to which the new dimension function is computed. We do not address in this paper the important question of determining the separable metric spaces X for which $\mathsf{dim}_{\mathsf{nbd}}(X)$ does not depend on the group into which X is homeomorphically embedded. At this point we do not have an example of an X for which this dependence occurs.

1. Selection principles, open covers and games

Let $(H, *)$ be a topological group with identity element e. We will assume that H is not compact. The collection of all open covers is denoted by $\mathcal{O}$.

The set $\mathcal{O}(U) = \{x * U : x \in H\}$ is an open cover of H. The symbol

$$\mathcal{O}_{nbd} = \{\mathcal{O}(U) : U \text{ a neighborhood of } e\}$$

denotes the collection of all such open covers of H.

Now we describe the relevant selection principles for this paper. Let S be an infinite set, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of S.
The selection principle $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is defined as follows:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n we have $B_n \in A_n$, and $\{B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

The selection principle $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$, introduced in [2], is defined as follows:

> For each sequence $(A_n : n < \infty)$ of elements of the family $\mathcal{A}$ there exists a sequence $(B_n : n < \infty)$ such that for each n B_n is a pairwise disjoint family refining A_n, and $\bigcup_{n<\infty} B_n$ is a member of the family $\mathcal{B}$.

We say that a space is *selectively screenable* if it has the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. The class of spaces satisfying $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ was introduced in [1]. We should point out that the selection principles $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ and $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ in topological groups do not coincide. In Theorem 1.2 of [13] E. and R. Pol use Martin's Axiom to construct a vector subspace M of the separable Hilbert space ℓ_2 such that the topological group $(M, +)$ has the property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ and the Menger property, but does not have the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. It can be shown that the topological groups $(\mathbb{R}^{\mathbb{N}}, +)$ and $(\mathsf{C}([0, 1], +)$ do not have the property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$.

The metrizable space X is said to be *Haver* [9] with respect to a metric d if there is for each sequence $(\epsilon_n : n < \infty)$ of positive reals a sequence $(\mathcal{V}_n : n < \infty)$ where each $\mathcal{V}_n$ is a pairwise disjoint family of open sets, each of d-diameter less than ϵ_n, such that $\bigcup_{n<\infty} \mathcal{V}_n$ is a cover of X.

A topological space X has the *Hurewicz property* if for each sequence $\mathcal{U}_n, n < \infty$ of open covers of X there is a sequence $\mathcal{F}_n, n < \infty$ of finite sets such that each $\mathcal{F}_n \subset \mathcal{U}_n$, and for each $x \in X$, the set $\{n : x \notin \cup\mathcal{F}_n\}$ is finite. We should point out that every σ-compact space has the Hurewicz property.

Let α be an ordinal number.

The game $\mathsf{G}_1^{\alpha}(\mathcal{A}, \mathcal{B})$ associated with the selection principle $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is as follows:

The players play an inning per $\gamma < \alpha$. In the γ-th inning ONE first chooses an $A_\gamma \in \mathcal{A}$: TWO then responds with a $B_\gamma \in A_\gamma$. A play $A_0, B_0, \cdots, A_\gamma, B_\gamma, \cdots$ of length α is won by TWO if $\{B_\gamma : \gamma < \alpha\} \in \mathcal{B}$. Else, ONE wins.

The game $\mathsf{G}_c^\alpha(\mathcal{A}, \mathcal{B})$ associated with the selection principle $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$ is as follows:

The players play an inning per $\gamma < \alpha$. In the γ-th inning ONE first chooses an $A_\gamma \in \mathcal{A}$: TWO then responds with a pairwise disjoint family of sets B_γ that refines A_γ. A play $A_0, B_0, \cdots, A_\gamma, B_\gamma, \cdots$ of length α is won by TWO if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$. Else, ONE wins.

When for a set S and families $\mathcal{A}$ and $\mathcal{B}$ there is an ordinal number α such that TWO has a winning strategy in the game $\mathsf{G}_c^\alpha(\mathcal{A}, \mathcal{B})$ played on S, then we define $\mathsf{tp}_{\mathsf{S}_c(\mathcal{A},\mathcal{B})}(S)$ to be $\min\{\alpha :$ TWO has a winning strategy in $\mathsf{G}_c^\alpha(\mathcal{A}, \mathcal{B})$ on $S\}$. We now define for topological group H the **neighborhood game dimension** of H, denoted $\mathsf{dim}_{\mathsf{nbd}}(H)$, by

$$1 + \mathsf{dim}_{\mathsf{nbd}}(H) = \mathsf{tp}_{\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})}(H)$$

The starting point for exploration of such game dimension of G is the following result from [4].

Theorem 1. *Let $(H, *)$ be a metrizable group. Then the following are equivalent:*

(1) $\mathsf{dim}(H) = n$.
(2) $\mathsf{dim}_{\mathsf{nbd}}(H) = n$.

We also have the following satisfying property of the neighborhood game dimension in the case of countable dimensional spaces:

Lemma 2 ([4])**.** *Let H be a metrizable group. The following are equivalent:*

(1) *H is countable dimensional.*
(2) $\mathsf{dim}_{\mathsf{nbd}}(H) = \omega$.

If for subspace X of the topological group H we define $\mathcal{O}_X$ to be the collection of covers of X by sets open in H, then the game $\mathsf{G}_c^\alpha(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}_X)$ is defined, as well as the ordinal function $\mathsf{tp}_{\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O}_X)}(H)$. Then we can defined $1 + \mathsf{dim}_{\mathsf{nbd}}(X) = \mathsf{tp}_{\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O}_X)}(H)$. The techniques for proving Theorem 1 and Lemma 2 apply also to this case: We find that for every finite dimensional space $X \subseteq H$, $\mathsf{dim}_{\mathsf{nbd}}(X) = \mathsf{dim}(X)$, and for each countable (and not finite) dimensional space $X \subseteq H$, $\mathsf{dim}_{\mathsf{nbd}}(X) = \omega$. Note that the concept $\mathsf{dim}_{\mathsf{nbd}}$ is defined relative to the group H. It is not clear how important the parameter H is. In this paper we assume H is $\mathbb{R}^{\mathbb{N}}$ or $\mathsf{C}[0, 1]$.

2. Selected examples and basic theorems.

A few examples from classical and recent literature are useful in illustrating some of the properties of the dimension function $\mathsf{dim}_{\mathsf{nbd}}$. For the reader's convenience we collect some of these here for reference further in the paper.

Example 1: (S. Mazurkiewicz) There exists a strongly infinite dimensional complete metric space $\mathbb{M}$ which is totally disconnected.

Example 2: (R. Pol) There is a compact metric space $\mathbb{K}$ of the form $\mathbb{L} \cup \mathbb{M}$ where $\mathbb{L}$ is a union of countably many compact finite dimensional spaces. ([15])

Example 3: (J. van Mill and R. Pol) There exists a complete metric space $\mathbb{V}$ with the properties that $\mathbb{V} = X \bigcup Y$ where X is countable dimensional and every $C \subseteq \mathbb{V} \setminus X$ which is closed in $\mathbb{V}$ is countable dimensional, but $\mathbb{V} \times \mathbb{V}$ is strongly infinite dimensional. ([11], Example 1.1)

Example 4: (E. Pol and R. Pol) There exists a complete metric space $(\mathbb{E}, d)$ such that $\mathbb{E} = X \bigcup Y$ where X is countable dimensional and every $C \subseteq \mathbb{E} \setminus X$ which is closed in $\mathbb{E}$ is countable dimensional, but $\mathbb{E} \times \mathbb{E}$ does not have the Haver property in the equivalent metric $\rho((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1),\ d(x_2, y_2)\}$. ([14])

Example 5: (E. Pol) There exists a separable metric space $\mathbb{F}$ of the form $X \bigcup Y$ such that X is countable dimensional and for any closed subset C of $\mathbb{F}$ such that $C \subseteq Y$, C is zerodimensional, and there is a zerodimensional subset B of the real line such that $\mathbb{F} \times B$ is strongly infinite dimensional. ([12], Example 1)

Example 6: (E. Pol and R. Pol) Assume Martin's Axiom. There is a vector subspace $\mathbb{G}$ of the separable Hilbert space ℓ_2 which has the property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ and the Menger property, but does not have the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. ([13], Theorem 1.2)

Example 7: $(\mathbb{R}^{\mathbb{N}}, +)$ is the group of sequences of real numbers with the operation of coordinatewise addition. Every separable metric space is homeomorphic to a subspace of this group.

Example 8: $(\mathsf{C}([0,1], +)$ is the group of continuous real-valued functions on the closed unit interval with the operation of pointwise addition. Every separable metric space embeds isometrically into this group.

The following theorems are among the fundamental tools we use to compute upper bounds for $\mathsf{dim}_{\mathsf{nbd}}$:

Theorem 3 (Lelek)**.** *If (X, d) is a complete metric space which is not compact, then it has a compactification $\mathsf{L}(X)$ of the form $X \cup C$ where C is a union of countably many compact finite dimensional spaces.*

Theorem 4 (Hurewicz-Tumarkin, [10])**.** *Let n be a non-negative integer. A separable metric space X is n-dimensional if, and only if, it is the union of $n+1$ but not fewer zero-dimensional subsets.*

Another basic tool is the following theorem (Theorem 2 of [4]) slightly adapted:

Theorem 5. *Let $(H, *)$ be a metrizable topological group and let X be a subspace of H. The following are equivalent:*

1. *$\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}_X)$ holds.*
2. *X has the Haver property in all left invariant metrics of $(H, *)$.*

The following fact is Theorem 5 of [3]:

Theorem 6. *For a subspace X of a metrizable Hurewicz space Y the following are equivalent:*

1. *$\mathsf{S}_c(\mathcal{O}, \mathcal{O}_X)$ holds*
2. *X has the Haver property in some equivalent metric of Y*
3. *X has the Haver property in all equivalent metrics of Y*

And the following fact is Theorem 5 of [4]:

Corollary 7. *For a subspace X of a metrizable Hurewicz group $(H, *)$ the following are equivalent:*

(1) $\mathsf{S}_c(\mathcal{O}, \mathcal{O}_X)$ *holds*
(2) $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}_X)$ *holds*

3. The relationship between the dimension functions $\mathsf{dim}_{\mathsf{G}}$ and $\mathsf{dim}_{\mathsf{nbd}}$

It is convenient that game dimension provides an upper bound for $\mathsf{dim}_{\mathsf{nbd}}$, since there are already some techniques (see [5]) for computing game dimension:

Theorem 8. *Let $(H, *)$ be a topological group and let $X \subseteq H$ be a subspace of H. Then*

$$\mathsf{dim}_{\mathsf{nbd}}(X) \leq \mathsf{dim}_{\mathsf{G}}(X) \leq \mathsf{dim}_{\mathsf{G}}(H).$$

Proof: Let σ be a winning strategy for TWO in the game $\mathsf{G}_c^{1+\alpha}(\mathcal{O}, \mathcal{O})$ played on X. Define a strategy τ for TWO in the game $\mathsf{G}_c^{1+\alpha}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}_X)$ as follows: For any open neighborhood U of the identity of H, consider the open cover $\mathcal{O}_X(U) = \{X \cap x * U : x \in G\}$ of X, and the corresponding open cover $\mathcal{O}(U) = \{x * U : x \in G\}$ of H.

To define $\tau(\mathcal{O}(U_1), \cdots, \mathcal{O}(U_n))$, compute in X the disjoint refinement

$$\sigma(\mathcal{O}_X(U_1), \cdots, \mathcal{O}_X(U_n))$$

of $\mathcal{O}_X(U_n)$. By Theorem II.21.1 on p. 226 of [10] there is a canonical disjoint family $\mathcal{F}$ of open sets in H such that $\{X \cap U : U \in \mathcal{F}\} = \sigma(\mathcal{O}_X(U_1), \cdots, \mathcal{O}_X(U_n))$. Define $\tau(\mathcal{O}(U_1), \cdots, \mathcal{O}(U_n)) = \mathcal{F}$.

Then τ is a winning strategy for TWO in the game $\mathsf{G}_c^{1+\alpha}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}_X)$. This establishes that $\mathsf{dim}_{\mathsf{nbd}}(X) \leq \mathsf{dim}_{\mathsf{G}}(X)$. The fact that $\mathsf{dim}_{\mathsf{G}}(X) \leq \mathsf{dim}_{\mathsf{G}}(H)$ follows from Theorem 11 below. $\square$

We shall see that the inequality between $\mathsf{dim}_{\mathsf{nbd}}$ and $\mathsf{dim}_{\mathsf{G}}$ can in fact be strict, even for completely metrizable spaces.

Corollary 9. *For any topological group $(H, *)$ which contains a homeomorphic copy of the space $\mathbb{K}$, we have $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{K}) = \omega + 1$.*

Proof: $\mathbb{K}$ is not countable dimensional, so by Theorem 15 of [4], $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{K}) > \omega$. Now apply Theorem 8 and the fact from [5] that $\mathsf{dim}_{\mathsf{G}}(\mathbb{K}) = \omega + 1$. $\square$

4. Monotonicity of $\mathsf{dim}_{\mathsf{nbd}}$.

A well-known classical theorem states

Theorem 10 (Monotonicity)**.** *If Y is a finite dimensional metric space and $X \subseteq Y$, then $\mathsf{dim}(X) \leq \mathsf{dim}(Y)$.*

This monotonicity theorem does not hold for the dimension function $\mathsf{dim}_{\mathsf{G}}$: The compact metric space $\mathbb{K}$ of Example 2 contains the strongly infinite dimensional subspace $\mathbb{M}$ of Example 1, and by the results of [5], $\mathsf{dim}_{\mathsf{G}}(\mathbb{K}) = \omega + 1$ while $\mathsf{dim}_{\mathsf{G}}(\mathbb{M}) = \omega_1$.

The monotonicity theorem holds for our new dimension function $\mathsf{dim}_{\mathsf{nbd}}$.

Theorem 11. *Let $(H, *)$ be a topological group and let $X \subseteq Y \subseteq H$ be subspaces of H. Then*

$$\mathsf{dim}_{\mathsf{nbd}}(X) \leq \mathsf{dim}_{\mathsf{nbd}}(Y) \leq \mathsf{dim}_{\mathsf{nbd}}(H).$$

Proof: Let α be the minimal ordinal such that TWO has a winning strategy in the game $\mathsf{G}_c^{1+\alpha}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ on H. Let σ be such a winning strategy for TWO. Then σ is also a winning strategy for TWO in the game $\mathsf{G}_c^{1+\alpha}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}_Y)$. It follows that the least β such that TWO has a winning strategy in $\mathsf{G}_c^{1+\beta}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}_Y)$ is no larger than α, and thus $\mathsf{dim}_{\mathsf{nbd}}(Y) \le \mathsf{dim}_{\mathsf{nbd}}(H)$. A similar argument shows that $\mathsf{dim}_{\mathsf{nbd}}(X) \le \mathsf{dim}_{\mathsf{nbd}}(Y)$. $\square$

The Mazurkiewicz space $\mathbb{M}$ of Example 1 is a subspace of $\mathbb{K}$, is complete and is strongly infinite dimensional. By Theorem 22 of [5] we have $\mathsf{dim}_{\mathsf{G}}(\mathbb{M}) = \omega_1$. But by Theorem 11 and Corollary 9, $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{M}) = \omega + 1$.

5. Products

Menger's product theorem ([10] Theorem II.VIII on p. 301) states:

Theorem 12. *For X and Y finite dimensional separable metric spaces,*

$$\mathsf{dim}(X \times Y) \le \mathsf{dim}(X) + \mathsf{dim}(Y).$$

Thus, when $\mathsf{dim}(Y) = 0$, we have $\mathsf{dim}(X \times Y) = \mathsf{dim}(X)$. This particular consequence of Menger's product theorem also holds for our new dimension function $\mathsf{dim}_{\mathsf{nbd}}$ as is shown below in Theorem 13. The analogue of Theorem 13 does not hold for game dimension $\mathsf{dim}_{\mathsf{G}}$. To see this consider Example 1 in [12]: Examining the details of this example we see that a separable metric space X and a set B of real numbers are constructed so that $\mathsf{dim}_{\mathsf{G}}(X) = \omega + 1$ and $\mathsf{dim}_{\mathsf{G}}(B) = 0$, but $\mathsf{dim}_{\mathsf{G}}(X \times B) = \omega_1$.

Theorem 13. *For $(H, *)$ and $(K, *)$ topological groups and $(K, *)$ zero-dimensional,*

$$\mathsf{dim}_{\mathsf{nbd}}(H \times K) = \mathsf{dim}_{\mathsf{nbd}}(H).$$

Proof: Fix an ordinal α with $\mathsf{dim}_{\mathsf{nbd}}(H) = \alpha$ and let σ be a winning strategy for TWO in the game $\mathsf{G}_c^{1+\alpha}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ on $(H, *)$.

Define a strategy τ of TWO as follows: When in inning γ ONE chooses a neighborhood $U_\gamma \times V_\gamma$, TWO applies strategy σ to find

$$\sigma(\mathcal{O}(U_\delta) : \delta \le \gamma),$$

a disjoint refinement of $\mathcal{O}(U_\alpha)$. Since H is zerodimensional, choose a disjoint refinement $\mathcal{H}_\gamma$ of $\mathcal{O}(V_\gamma)$ such that $\mathcal{H}_\gamma$ covers H. Define

$$\tau(\mathcal{O}(U_\delta \times V_\delta) : \delta \le \gamma) := \{A \times B : A \in \sigma(\mathcal{O}(U_\delta) : \delta \le \gamma) \text{ and } B \in \mathcal{H}_\gamma\}$$

It is clear that $\tau(\mathcal{O}(U_\delta \times V_\delta) : \delta \le \gamma)$ is a disjoint family of open sets. We must see that it refines $\mathcal{O}(U_\gamma \times V_\gamma)$, and that TWO wins each play of length $1 + \alpha$ played using τ.

Let $A \times B \in \tau(\mathcal{O}(U_\delta \times V_\delta) : \delta \le \gamma)$ be given. Choose an $x \in G$ with $A \subseteq x * U_\gamma$, and choose a $y \in H$ with $B \subseteq y * V_\gamma$. Then $A \times B \subseteq (x, y) * U_\gamma \times V_\gamma$ and the latter is an element of $\mathcal{O}(U_\gamma \times V_\gamma)$.

To see that TWO wins a play of length $1 + \alpha$, consider an $(x, y) \in G \times H$. Since σ is a winning strategy for TWO in $\mathsf{G}_c^{1+\alpha}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ on H, find a $\gamma < 1 + \alpha$ with $x \in \bigcup \sigma(\mathcal{O}(U_\delta) : \delta \le \gamma)$, and choose $A \in \sigma(\mathcal{O}(U_\delta) : \delta \le \gamma)$ with $x \in A$. Since $\mathcal{H}_\gamma$ covers H, find a $B \in \mathcal{H}_\gamma$ with $y \in B$. Then we have

$$(x, y) \in A \times B \in \tau(\mathcal{O}(U_\delta \times V_\delta) : \delta \le \gamma). \qquad \square$$

More generally, the following analogues of Menger's Product Theorem hold for the dimension function $\mathsf{dim}_{\mathsf{nbd}}$:

Theorem 14. *For $(H, *)$ and $(K, *)$ topological groups and $(K, *)$ finite dimensional, and $\mathsf{dim}_{\mathsf{nbd}}(H) = \alpha + n$ where α is a limit ordinal and $0 \le n < \omega$,*

$$\mathsf{dim}_{\mathsf{nbd}}(H \times K) \leq \mathsf{dim}_{\mathsf{nbd}}(H) + \mathsf{dim}_{\mathsf{nbd}}(K) \cdot n.$$

Proof: Consider the case when $n > 0$. Let the dimension of K be k. Then write $K = \bigcup_{j=1}^{k+1} K_j$ where each K_j is zero-dimensional. Also, write $\alpha = \bigcup_{j=1}^{k+1} S_j$ where each S_j has order type α. For each j let σ_j be a winning strategy for TWO in the game $\mathsf{G}_c^{\alpha+n}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ played on $H \times K_j$ as in Theorem 13. Now play the first α innings as follows: In inning $\gamma < \alpha$ first identify j with $\gamma \in S_j$. Then put

$$\tau(\mathcal{U}_\nu : \nu \leq \gamma) = \sigma_j(\mathcal{U}_\mu : \mu \leq \gamma \text{ and } \mu \in S_j).$$

After α innings, TWO has completed α innings in each of the games associated with $H \times K_j$, using a winning strategy for each of the corresponding games. Thus, for each j, the uncovered part of $H \times K_j$ has dimension less than or equal to n. By Theorem 4 the union of these k sets has dimension at most $k \cdot n - 1 = \dim(H) \cdot n - 1$. By Theorem 15 of [4] covering this requires no more than $k \cdot n$ additional innings. □

Analogous arguments can be used to prove the following two theorems.

Theorem 15. *For $(H, *)$ and $(K, *)$ topological groups and $(K, *)$ countable dimensional, and $\mathsf{dim}_{\mathsf{nbd}}(H)$ a successor ordinal,*

$$\mathsf{dim}_{\mathsf{nbd}}(H \times K) \leq \mathsf{dim}_{\mathsf{nbd}}(H) + \mathsf{dim}_{\mathsf{nbd}}(K).$$

Theorem 16. *For $(H, *)$ and $(K, *)$ topological groups and $(K, *)$ countable dimensional, and $\mathsf{dim}_{\mathsf{nbd}}(H)$ a limit ordinal,*

$$\mathsf{dim}_{\mathsf{nbd}}(H \times K) = \mathsf{dim}_{\mathsf{nbd}}(H).$$

6. The union theorem

For finite dimensional metric spaces we have the following classical theorem:

Theorem 17 (Union Theorem)**.** *If X and Y are finite dimensional subspaces of a metric space Z then*

$$\mathsf{dim}(X \cup Y) \leq \mathsf{dim}(X) + \mathsf{dim}(Y) + 1.$$

A corresponding theorem holds for $\mathsf{dim}_{\mathsf{nbd}}$. First we define, as in [6], for ordinals α and β the "sum" $\alpha \oplus \beta$ as follows: Write $\alpha = \alpha' + m$ and $\beta = \beta' + n$ where α' and β' are limit ordinals or zero and $m, n < \omega$.

$$\alpha \oplus \beta = \begin{cases} \alpha & \text{if } \alpha' > \beta' \\ \alpha + n & \text{if } \alpha' = \beta' \\ \beta & \text{otherwise} \end{cases}$$

Theorem 18 (Union Theorem)**.** *If X and Y are subspaces of a separable metric space Z then*

$$\mathsf{dim}_{\mathsf{nbd}}(X \cup Y) \leq \mathsf{dim}_{\mathsf{nbd}}(X) \oplus \mathsf{dim}_{\mathsf{nbd}}(Y).$$

Proof: Choose winning strategies σ_X and σ_Y for TWO in the corresponding games of the corresponding lengths. If $\alpha' > \beta'$, then write $\beta' = S_1 \cup S_2$ where S_1 and S_2 are disjoint sets, each of order type β'. During the first β' innings, if the inning number is in S_1, TWO plays the strategy σ_X, and if the inning number is in S_2, TWO plays σ_Y. After these innings TWO plays σ_Y for the next finite number of innings until Y is covered, and then plays σ_X the rest of the innings until X

is also covered. This takes α innings in total. A similar argument shows that if $\alpha' < \beta'$, then TWO wins the game on $X \cup Y$ in β innings. Thus assume that $\alpha' = \beta'$. Then using the above schedule based on S_1 and S_2 we see that after α' innings the uncovered part of X is $m-1$-dimensional, and the uncovered part of Y is $n-1$-dimensional. By Theorem 17 the dimension of the uncovered part of $X \cup Y$ is at most $m+n-1$. Thus, covering this part requires at most $m+n$ additional innings. $\square$

7. Discussion of examples.

In the analysis of examples below we are using the following general fact:

Theorem 19. *Let X be a completely metrizable separable metric space with compactification $\mathsf{L}(X)$ as in Theorem 3. Assume that for each natural number n we have $\mathsf{dim}_{\mathsf{G}}(\mathsf{L}(X))^n) \leq \alpha_n$. Then for the subgroup $\langle \mathsf{L}(X) \rangle$ of $\mathsf{C}[0,1]$ generated by an isometric copy of $\mathsf{L}(X)$, $\mathsf{dim}_{\mathsf{G}}(\langle \mathsf{L}(X) \rangle) \leq \alpha$, where $\alpha = \sup\{\alpha_n : n < \omega\}$.*

Example 1: $\mathbb{M}$

By Theorem 8 and the fact that $\mathbb{M}$ is a subspace of $\mathbb{K}$ we have $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{M}) = \omega + 1$, and $\mathsf{dim}_{\mathsf{nbd}}(\langle \mathbb{M} \rangle) \leq \omega^2$. But since $\mathbb{M}$ is strongly infinite dimensional, $\mathsf{dim}_{\mathsf{G}}(\mathbb{M}) = \omega_1$.

Example 2: $\mathbb{K}$

As noted earlier, $\mathsf{dim}_{\mathsf{G}}(\mathbb{K}) = \omega + 1 = \mathsf{dim}_{\mathsf{nbd}}(\mathbb{K})$. It was shown in [5] that for each n $\dim(\mathbb{K}^n) \leq \omega \cdot n + 1$, and thus by Theorem 8, $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{K}^n) \leq \omega \cdot n + 1$. Then Theorem 19 implies that $\mathsf{dim}_{\mathsf{nbd}}(\langle \mathbb{K} \rangle) \leq \omega^2$.

Example 3: $\mathbb{V}$

Examination of Example 1.1 in [11] shows that J. van Mill and R. Pol constructed a complete separable metric space space $\mathbb{V}$ such that $\mathsf{dim}_{\mathsf{G}}(\mathbb{V}) \leq \omega \cdot 2$, and yet $\mathsf{dim}_{\mathsf{G}}(\mathbb{V} \times \mathbb{V}) = \omega_1$. The space $\mathbb{V}$ has the following property:

$$\mathbb{V} = X \bigcup Y$$

where X is countable dimensional, and for any closed subset C of $\mathbb{V}$ disjoint from X, C is countable dimensional. Now let $\mathsf{L}(\mathbb{V})$ be a compactification of $\mathbb{V}$ as in Theorem 3. Then still we have $\mathsf{dim}_{\mathsf{G}}(\mathsf{L}(\mathbb{V})) \leq \omega \cdot 2$.

Lemma 20. *For each positive integer n, $\mathsf{dim}_{\mathsf{G}}(\mathsf{L}(\mathbb{V})^n) \leq \omega \cdot (n+1)$.*

Proof: The proof is by induction on n. For $n = 1$ we are done. Thus, assume that $n > 1$ and that for all $k < n$ we have already verified that $\mathsf{dim}_{\mathsf{G}}(\mathsf{L}(\mathbb{V})^k) \leq \omega \cdot k$. Choose k so that $n = k+1$ and consider the space $\mathsf{L}(\mathbb{V})^{k+1}$. Since $(L \bigcup X)^n$ is countable dimensional, during the first ω innings of the game, TWO covers this part. The complement of what TWO has covered is compact, being a closed subset of a compact space. The projection of this compact set on each coordinate axis is compact, and a subset of Y, thus compact and countable dimensional. Let C be the union of these n projections. By the induction hypothesis $\mathsf{dim}_{\mathsf{G}}(\mathsf{L}(\mathbb{V})^{n-1}) \leq \omega \cdot n$. But then as $\mathsf{L}(\mathbb{V})^{n-1}$ is compact, Theorem 32 of [5] implies that $\mathsf{dim}_{\mathsf{G}}(C \times \mathsf{L}(\mathbb{V})^{n-1}) \leq \omega \cdot n$. Now during the next $\omega \cdot n$ innings TWO covers the rest of the space $\mathsf{L}(\mathbb{V})^n$. $\square$

It follows that the subgroup $\langle \mathsf{L}(\mathbb{V}) \rangle$ of $\mathsf{C}[0,1]$ has $\mathsf{dim}_{\mathsf{G}}(\langle \mathsf{L}(\mathbb{V}) \rangle) \leq \omega^2$. Applying Theorems 8 and 19 we find that for each n, $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{V}^n) \leq \omega \cdot (n+1)$, even though $\mathsf{dim}_{\mathsf{G}}(\mathbb{V}^2) = \omega_1$, and also that $\mathsf{dim}_{\mathsf{nbd}}(\langle \mathsf{L}(\mathbb{V}) \rangle) \leq \omega^2$.

Example 4: $\mathbb{E}$

By similar arguments we find that $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{E}^n) \leq \omega \cdot (n+1)$ for each positive integer n. Recall that there is a metric on $\mathbb{E}^2$ in which it does not have the Haver property. But $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{E}^n) \leq \omega \cdot (n+1)$ implies that $\mathbb{E}^n$ is Haver bounded in all translation invariant metrics of $\mathsf{C}[0,1]$. Thus, the metric in which the square of $\mathbb{E}$ does not have the Haver property does not extend to a translation invariant metric on $\mathsf{C}[0,1]$.

Example 5: $\mathbb{F}$

Examination of this example shows that $\mathsf{dim}_{\mathsf{G}}(\mathbb{F}) = \omega + 1$. By Theorem 8 $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{F}) \leq \omega + 1$. But since $\mathbb{F}$ is not countable dimensional, we have $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{F}) = \omega + 1$. Then by Theorem 13, for any zero-dimensional metric space B, $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{F} \times B) = \omega + 1$. But there is a zero-dimensional set B of real numbers for which $\mathsf{dim}_{\mathsf{G}}(\mathbb{F} \times B) = \omega_1$.

$\mathbb{F}$ is in fact a subspace of $\mathbb{C} \times \mathbb{K}$, where $\mathbb{C}$ is the Cantor discontinuum. Since for each positive integer n, $\mathbb{C}^n$ is homeomorphic to $\mathbb{C}$, results of [5] imply that $\mathsf{dim}_{\mathsf{G}}((\mathbb{K} \times \mathbb{C})^n) = \mathsf{dim}_{\mathsf{G}}(\mathbb{K}^n) \leq \omega \cdot n + 1$. But then we have by Theorem 19 that $\mathsf{dim}_{\mathsf{G}}(\langle \mathbb{K} \times \mathbb{C} \rangle) \leq \omega^2$. It follows from Theorem 8 that $\mathsf{dim}_{\mathsf{nbd}}(\langle \mathbb{F} \rangle) \leq \omega^2$, and for each n, that $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{F}^n) \leq \omega \cdot n + 1$.

Example 6: $\mathbb{G}$

The proof of (17) of [13] shows that for each space A of cardinality less than $2^{\aleph_0}$, the space $A \times T$ has $\mathsf{dim}_{\mathsf{G}}(A \times T) = \omega + 1$. A rough analysis of the space T constructed for Proposition 5.1 of [13] reveals that for each positive integer n we have $\mathsf{dim}_{\mathsf{G}}(T^n) \leq \omega^\omega$ (the least countable ordinal that exceeds ω^n for each n).

Thus there is a vector subspace $\langle T \rangle$ of ℓ_2 such that $\mathsf{dim}_{\mathsf{G}}(\langle T \rangle) \leq \omega^\omega$, and T is homeomorphic to a closed subspace of $\langle T \rangle$. It follows that the Menger subspace S of T constructed for Proposition 5.1 of [13] generates a vector subspace $\mathbb{G} = \langle S \rangle$ of $\langle T \rangle$ which still has the Menger property, and which contains a homeomorphic copy of S as a closed subspace. By Theorem 8, $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{G}) \leq \omega^\omega$, and yet $\mathsf{dim}_{\mathsf{G}}(\mathbb{G}) = \omega_1$.

Example 7: $\mathbb{R}^{\mathbb{N}}$

The neighborhood dimension $\mathsf{dim}_{\mathsf{nbd}}(\mathbb{R}^{\mathbb{N}}) = \omega_1$. The space $\mathbb{R}^{\mathbb{N}}$ is hereditarily Lindelöf. By a theorem of P. Daniels and G. Gruenhage [8] TWO wins the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ and so TWO wins the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}_{nbd}, \mathcal{O})$ which is a special case of $\mathsf{G}_c^{\omega_1}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$. But, TWO can not win the game $\mathsf{G}_c^{\alpha}(\mathcal{O}_{nbd}, \mathcal{O})$ in fewer than ω_1 innings since $(\mathbb{R}^{\mathbb{N}}, +)$ is not $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$. To see this, recall that the group generated by the Hilbert cube, $[0,1]^{\mathbb{N}}$, is σ-compact and consequently has the Hurewicz property. It is easy to see that any subgroup of a group with $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$ has the $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$ property. Now, suppose $(\mathbb{R}^{\mathbb{N}}, +)$ has $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$. Then the group generated by $[0,1]^{\mathbb{N}}$ as a subgroup of subgroup of $(\mathbb{R}^{\mathbb{N}}, +)$ also has $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$ and by Theorem 5 of [5] we have that it must be selectively screenable. But by a theorem of J. Nagata, the Hilbert cube is strongly infinite dimensional and so it cannot be selectively screenable.

Considered as vector space the algebraic dimension of $\mathbb{R}^{\mathbb{N}}$, written $\mathsf{dim}_{\mathsf{alg}}(\mathbb{R}^{\mathbb{N}})$, is $2^{\aleph_0}$. We have for each finite n that

$$\mathsf{dim}_{\mathsf{nbd}}(\mathbb{R}^n) = n = \mathsf{dim}_{\mathsf{alg}}(\mathbb{R}^n).$$

The equation

$$\mathsf{dim}_{\mathsf{nbd}}(\mathbb{R}^{\mathbb{N}}) = \mathsf{dim}_{\mathsf{alg}}(\mathbb{R}^{\mathbb{N}})$$

is equivalent to the Continuum Hypothesis.

Example 8: $\mathsf{C}[0,1]$

Also $\mathsf{dim}_{\mathsf{nbd}}(\mathsf{C}[0,1]) = \omega_1$ and $\mathsf{dim}_{\mathsf{alg}}(\mathsf{C}[0,1]) = 2^{\aleph_0}$, and so the Continuum Hypothesis is equivalent to the statement that $\mathsf{dim}_{\mathsf{nbd}}(\mathsf{C}[0,1]) = \mathsf{dim}_{\mathsf{alg}}(\mathsf{C}[0,1])$

8. Remarks

As the reader may have noticed, for the infinite dimensional spaces for which we have information about $\mathsf{dim}_{\mathsf{nbd}}$, this information is mostly in terms of upper bounds, with the obvious lower bound being $\omega+1$. We suspect that our upper bounds are in fact sharp.

Example X	$\mathsf{dim}_{\mathsf{nbd}}(X)$	$\mathsf{dim}_{\mathsf{G}}(X)$
$\mathbb{M}$	$\omega+1$	ω_1
$\langle\mathbb{M}\rangle$	$\leq \omega^2$	ω_1
$\mathbb{K}$	$\omega+1$	$\omega+1$
$\mathbb{K}^n$	$\leq \omega\cdot n+1$	$\leq \omega\cdot n+1$
$\langle\mathbb{K}\rangle$	$\leq \omega^2$	$\leq \omega^2$
$\mathbb{V}$	$\leq \omega\cdot 2$	$\leq \omega\cdot 2$
$\mathbb{V}^n$	$\leq \omega\cdot(n+1)$	ω_1, $n>1$
$\langle\mathbb{V}\rangle$	$\leq \omega^2$	ω_1
$\mathbb{E}^n$	$\leq \omega\cdot(n+1)$	$\leq \omega\cdot(n+1)$
$\mathbb{F}$	$\leq \omega+1$	$\omega+1$
$\langle\mathbb{F}\rangle$	$\leq \omega^2$	ω_1
$\mathbb{G}$	$\leq \omega^\omega$	ω_1
$\mathbb{R}^{\mathbb{N}}$	ω_1	ω_1
$\mathsf{C}[0,1])$	ω_1	ω_1

We have also not explored the question whether $\mathsf{dim}_{\mathsf{nbd}}$ depends on the ambient group in which it is being computed. It seems important to know for which spaces X the dimension $\mathsf{dim}_{\mathsf{nbd}}(X)$ depends on the group into which X is embedded as subspace.

Question 1: Is $\mathsf{dim}_{\mathsf{nbd}}(X)$ independent of the group into which X is embedded as subspace?

There are other well-known dimension functions that were introduced for compact metrizable spaces by R. Pol [16] and by P. Borst [6] and [7]. These dimension functions assign ordinals larger than ω to some countable dimensional spaces, but it appears that these have not been computed for even classical examples such as $\mathbb{K}$. It is not known how $\mathsf{dim}_{\mathsf{nbd}}$ is related to the dimension functions of Pol and of Borst.

Question 2: How does $\mathsf{dim}_{\mathsf{nbd}}(X)$ compare with other well-known dimension functions?

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