Relative Hurewicz property and games

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Abstract

In this paper we assume that X is *Lindelöf* space. Our main results characterzes the relative Hurewicz property game-theoretically.
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Let X be a topological space and let Y be a subspace of X, possibly equal to X. By a cover for X we always mean "countable open cover". Since we are primarily interested in Lindelof spaces, the restriction to countable covers does not lead to a loss of generality. We will use the following classes of open covers:

1. Γ – the collection of γ-covers of the space. An open cover $\mathcal{U}$ of X is said to be a γ-*cover* if it is infinite and for each $x \in X$ the set $\{U \in \mathcal{U} : x \notin U\}$ is finite.

2. Λ – the collection of λ-covers of the space. An open cover $\mathcal{U}$ of X is said to be a λ-*cover* cover if: for each $x \in X$ the set $\{U \in \mathcal{U} : x \in U\}$ is infinite.

The symbol Γ_X (Λ_X) denotes the set of γ-covers (λ-covers) of X, and Γ_Y (Λ_Y) denotes the collection of γ-covers (λ-covers) of Y by sets open in X.

In [3] Hurewicz introduced a covering property which is nowadays called the Hurewicz property: For each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of X there is a sequence $(\mathcal{V}_n : n < \infty)$ of finite sets such that for each n $\mathcal{V}_n \subseteq \mathcal{U}_n$, and each element of X belongs to all but finitely many of the sets $\cup\mathcal{V}_n$.
The symbol $\mathsf{U}_{fin}(\mathcal{A}_X, \mathcal{B}_Y)$ denotes the following selection principle: For a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements in $\mathcal{A}_X$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, U_n is a finite subset of $\mathcal{U}_n$ and $\{\bigcup U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}_Y$, or there exist an n such that $Y \subseteq \bigcup U_n$. According to [7] we say that Y has the relative Hurewicz property in X if the selection principle $\mathsf{U}_{fin}(\Gamma_X, \Gamma_Y)$ holds.

Theorem 1 $\mathsf{U}_{fin}(\Gamma_X, \Gamma_Y) = \mathsf{U}_{fin}(\Lambda_X, \Gamma_Y)$.

Proof : The implication $\mathsf{U}_{fin}(\Lambda_X, \Gamma_Y) \Rightarrow \mathsf{U}_{fin}(\Gamma_X, \Gamma_Y)$ is evident. Now, let $\mathsf{U}_{fin}(\Gamma_X, \Gamma_Y)$ hold and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ-covers of X. We may assume that for each finite subset $\mathcal{F} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ we have that $\mathcal{U}_k \cap \mathcal{F} = \emptyset$ for all but finitely many k.

For each n, enumerate $\mathcal{U}_n$ bijectively as $(U^n_k : k \in \mathbb{N})$ and define

$$V^n_m = \bigcup\{U^n_i : i < m\}.$$

We have that each $\mathcal{V}_n = (V^n_m : m \in \mathbb{N})$ is open cover of X for which: either there is exists m_n, such that $V^n_{m_n} = X$, or $\mathcal{V}_n$ is a γ-cover of X. We must consider the case where there exists an infinite set A for which $\mathcal{V}_n$ is a γ-cover for each $n \in A$. So, let $\mathcal{V}_n$ be a γ-cover for each $n \in A$. We apply the selection principle $\mathsf{U}_{fin}(\Gamma_X, \Gamma_Y)$ and we can find a finite subset $\mathcal{W}_n \subset \mathcal{V}_n$ such that $\bigcup \mathcal{W}_n : n \in A$ is γ-cover of Y in X. $\diamondsuit$

We define the following game associated to $\mathsf{U}_{fin}(\Gamma_X, \Gamma_Y)$: in the n-th inning player ONE selects a γ-cover $\mathcal{U}_n$; player TWO responds by selecting a finite set $U_n \subset \mathcal{U}_n$. TWO wins the play $\mathcal{U}_1, U_1; \mathcal{U}_2, U_2; \ldots, \mathcal{U}_n, U_n; \ldots$ if $\{\bigcup U_n : n \in \mathbb{N}\} \in \Gamma_Y$; otherwise ONE wins. This game is denoted $\mathsf{Hurewicz}(X, Y)$.

Theorem 2 *For a Lindelöf space X and $Y \subset X$ the following are equivalent:*

1. *The $\mathsf{U}_{fin}(\Gamma_X, \Gamma_Y)$ property hold;*
2. *ONE does not have a winning strategy in the game $\mathsf{Hurewicz}(X, Y)$.*

Proof : (1) $\Rightarrow$ (2): Let F be a strategy for the player ONE. The first move of the player ONE according to the strategy F will be denoted with $F(X)$. We will prove that for the strategy F of player ONE there exist a play of the game

$$F(X), T_1 \subset F(X), F(T_1), T_2 \subset F(T_1), F(T_1, T_2), T_3 \subset F(T_1, T_2), \cdots$$

where $T_1, T_2, T_3, \cdots$ are finite sets such that the player TWO wins i.e.

$$(\forall y \in Y)(\forall^\infty_n)^1(y \in \cup T_n).$$

Because X is Lindelöf we may assume that the player ONE always chooses a countable λ-cover for the space X. For each λ-cover $(U_n : n \in \mathbb{N})$ for X which player ONE chooses, the counter strategy of player TWO is to choose only sets of the form $\{U_1\}, \{U_1, U_2\}, \cdots, \{U_1, U_2, \cdots, U_n\}, \cdots$.

Then the moves of player ONE according to the startegy F in the game $\mathsf{Hurewicz}(X, Y)$ are as follows:
For each n_1

$$\mathcal{U}_{n_1} = F(\{U_1, U_2, \cdots, U_{n_1}\})$$

enumerate this λ-cover as $\mathcal{U}_{n_1} = (U_{n_1,n} : n \in \mathbb{N})$;
For each n_1, n_2

$$\mathcal{U}_{(n_1,n_2)} = F(\{U_1, U_2, \cdots, U_{n_1}\}, \{U_{n_1,1}, U_{n_1,2}, \cdots, U_{n_1,n_2}\})$$

[1] for all but finitely many

enumerate this λ-cover as $\mathcal{U}_{(n_1,n_2)} = (U_{n_1,n_2,n} : n \in \mathbb{N})$;

$\vdots$

$\vdots$

For each $n_1, n_2, \cdots, n_k$,

$$\mathcal{U}_{n_1,n_2,\cdots,n_k} = F(\{U_1, \cdots, U_{n_1}\}, \{U_{n_1,1}, \cdots, U_{n_1,n_2}\}, \{U_{n_{k-1,1}}, \cdots, U_{n_1,n_2,\cdots,n_k}\})$$

enumerate this λ-cover as $\mathcal{U}_{(n_1 n_2 \cdots n_k)} = (U_{n_1,n_2,\cdots,n_k,n} : n \in \mathbb{N})$;

In this way we get a countable family of λ-covers for X:

$$(\mathcal{U}_{(n_1,n_2\cdots,n_k)} : n_1, n_2, \cdots, n_k \in \mathbb{N}, k < \mathbb{N}).$$

Since Y has *Hurewicz* property in X: for each $(n_1, n_2, \cdots, n_k)$ we choose a finite set $\mathcal{V}_{(n_1,n_2,\cdots,n_k)} \subset \mathcal{U}_{(n_1,n_2,\cdots,n_k)}$, such that

$$(\forall y \in Y)(\forall^{\infty}_{(n_1,n_2,\cdots,n_k)})(y \in \cup \mathcal{V}_{(n_1,n_2,\cdots,n_k)}).$$

For each $\mathcal{V}_{(n_1,n_2,\cdots,n_k)} \subseteq \mathcal{U}_{(n_1,n_2,\cdots,n_k)}$ we choose n_{k+1}, such that

$$\mathcal{V}_{(n_1,n_2,\cdots,n_k)} \subset \{U_{n_1,\cdots,n_k}, \cdots, U_{n_1,\cdots,n_k,n_{k+1}}\} = T_{k+1}$$

The sequence of moves $F(X), T_1, F(T_1), T_2, F(T_1, T_2), T_3, \cdots$ of players ONE and TWO in the game $\mathsf{Hurewicz}(X, Y)$ is according to the strategy F of player ONE such that for each k, we have

$$\cup \mathcal{V}_{(n_1,n_2,\cdots,n_k)} \subseteq \cup T_{k+1}.$$

Now $\mathcal{V}_{(n_1)}, \mathcal{V}_{(n_1,n_2)}, \cdots, \mathcal{V}_{(n_1,n_2,\cdots,n_k)} \cdots$ is an infinite subset of the set of all $\mathcal{V}_\sigma$ (where σ is any finite sequence of natural numbers). We have that

$$(\forall y \in Y)(\forall^{\infty}_{k})(y \in \cup \mathcal{V}_{(n_1,n_2,\cdots,n_k)} \subseteq \cup T_{k+1}).$$

Consequently F is not a winning strategy for player ONE. $\diamondsuit$

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