



Combinatorics of open covers (VIII)

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Abstract

For each space, $U_{\text{fin}}(\Gamma, \Omega)$ is equivalent to $S_{\text{fin}}(\Omega, \mathcal{O}^{wgp})$ and this selection property has game-theoretic and Ramsey-theoretic characterizations (Theorem 2). For Lindelöf space X we characterize when a subspace Y is relatively Hurewicz in X in terms of selection principles (Theorem 9), and for metrizable X in terms of basis properties, and measurelike properties (Theorems 14 and 16). Using the Continuum Hypothesis we show that there is a subset Y of the Cantor set \mathbf{C} which has the relative γ -property in \mathbf{C} , but Y does not have the Menger property.

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Introduction

In this paper we continue previous investigations into the combinatorial properties of sequences of open covers of topological spaces. Let \mathcal{A} and \mathcal{B} be collections of subsets of an infinite set S .

The symbol $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denotes the statement that there is for each sequence $(O_n: n \in \mathbb{N})$ of elements of \mathcal{A} , a sequence $(T_n: n \in \mathbb{N})$ of finite sets such that for each n we have $T_n \subset O_n$, and $\bigcup_{n \in \mathbb{N}} T_n \in \mathcal{B}$. In [9] Hurewicz considered this statement for the case when both \mathcal{A} and \mathcal{B} are the collection of all open covers of a given topological space. Hurewicz proved that for metric spaces this selection hypothesis is equivalent to a property introduced a year earlier by Menger. Nowadays, when a space satisfies Hurewicz's selection hypothesis, it is said to have the *Menger property*.

The symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the statement that there is for each sequence $(O_n: n \in \mathbb{N})$ of elements of \mathcal{A} , a sequence $(T_n: n \in \mathbb{N})$ such that for each n we have $T_n \in O_n$, and $\{T_n: n \in \mathbb{N}\} \in \mathcal{B}$. In [16] Rothberger considered this statement for the case when both \mathcal{A} and \mathcal{B} are the collection of all open covers of a given topological space. He showed that this property implies the strong measure zero property introduced in 1919 by Borel. When a space satisfies Rothberger's selection hypothesis, it is said to have the *Rothberger property*.

Several other examples of mathematical properties defined, or characterized, in terms of these two selection principles, exist in current literature. Perhaps the best known among these because of their role in function space theory, are Arkhangel'skii's countable fan tightness property, the Fréchet–Urysohn property, the γ -property of Gerlits and Nagy, the Sakai property, and yet another Gerlits–Nagy property. These terms will be defined later on in the paper as needed. But also well known are examples of these properties in the theory of the Stone–Čech compactification of the set of natural numbers: The so-called P -points have a characterization in terms of an S_{fin} -type selection principle, while selective ultrafilters have a characterization in terms of an S_1 -type selection principle.

In the earlier papers in this *Combinatorics of Open Covers* series the selection principles $S_1(\mathcal{A}, \mathcal{B})$ and $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ have been extensively studied for the case when \mathcal{A} and \mathcal{B} were for a fixed space X families of open covers with special properties. Recently the following interesting generalization of these works was introduced in [11]: Let spaces X and Y be given with Y a subspace of X . Consider these selection principles when \mathcal{A} is a family of open covers of the superspace X , and \mathcal{B} is a family of covers of Y by sets open in X . By hindsight, some earlier results—as in [18], for example—characterized some important classical properties in terms of this scenario.

In the case when these selection properties are for open covers of one fixed space X , we shall call them *absolute selection principles*. In the other case, where they refer to covers of a subspace Y of some superspace X by sets open in X , and selection is from covers of X , we shall call them *relative selection principles*.

Part of the interest in these two selection principles stem from the fact that they seem to be the key in establishing a connection between the subjects where they arise, with another area of mathematics which arose independently from a 1930 result of F.P. Ramsey, and subsequent intensive activity by Erdős and numerous collaborators and contributors to the area—Ramsey theory. Thus far the main technical tools used to establish this connection

have been infinite games. In [12] much of these techniques has been given a general framework for easy applicability.

The paper is organized to have a background section, followed by four parts. In the Background section we familiarize the reader with the Ramsey-theoretic statements, the game-theoretic statements and notation of the paper. Then Part I is devoted to the study of two specific absolute selection principles. We show that a selection principle introduced in an earlier paper, and which appeared to not be an S_1 - or S_{fin} -type selection principle, is really an S_{fin} -principle. The main innovation needed was to introduce an appropriate class of open covers for spaces—the so-called *weakly groupable* open covers. This, plus the results in [12], show that all the selection principles that were considered in [17] and in [10] are of the S_1 - or the S_{fin} -kind. This new information may be useful in solving Problems 1 or 2 of [10].

In Part II we characterize the relative Hurewicz property in a manner analogous to the characterization of the absolute Hurewicz property given in [12]. In Part III we turn attention specifically to metrizable spaces, and characterize the relative Hurewicz property there in terms of basis theory, and metric measure theory. In Part IV we show that the relative selection properties are much different from their absolute counterparts by showing that the Continuum Hypothesis implies the existence of a relative gamma set which does not have the absolute Menger property, and thus is not a γ -set.

Background

Ramsey theory

For positive integers n and k the symbol $\mathcal{A} \rightarrow (\mathcal{B})_k^n$ denotes the statement:

For each $A \in \mathcal{A}$ and for each function $f : [A]^n \rightarrow \{1, \dots, k\}$ there are a set $B \subseteq A$ and a $j \in \{1, \dots, k\}$ such that for each $Y \in [B]^n$, $f(Y) = j$, and $B \in \mathcal{B}$.

Here $[A]^n$ denotes the set of n -element subsets of A . We call f a “coloring”, and we say that “ B is homogeneous of color j for f ”.

This symbol is called the *ordinary partition symbol*. It is one of many “partition symbols” that have been extensively studied. Ramsey theory deals with partition symbols. F.P. Ramsey proved the first important partition theorem. The ordinary partition symbol denotes a relation between \mathcal{A} and \mathcal{B} , and this relation is called the *ordinary partition relation*. Several selection principles of the form $S_1(\mathcal{A}, \mathcal{B})$ have been characterized by the ordinary partition relation.

Another partition symbol important for the study of selection principles is motivated by a study of Baumgartner and Taylor in [3]. For each positive integer k , $\mathcal{A} \rightarrow [\mathcal{B}]_k^2$ denotes the following statement:

For each A in \mathcal{A} and for each function $f : [A]^2 \rightarrow \{1, \dots, k\}$ there is a set $B \subset A$ and a $j \in \{1, \dots, k\}$, and a partition $B = \bigcup_{n \in \mathbb{N}} B_n$ of B into pairwise disjoint finite sets

such that for each $\{a, b\} \in [B]^2$ for which a and b are not from the same B_n , we have $f(\{a, b\}) = j$, and $B \in \mathcal{B}$.

We say that “ B is nearly homogeneous for f ”. The relation between \mathcal{A} and \mathcal{B} denoted by this partition symbol is called the *Baumgartner–Taylor partition relation*. Several selection principles of the form $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ have been characterized by the Baumgartner–Taylor partition relation.

When the pair $(\mathcal{A}, \mathcal{B})$ are appropriately related, one can show that if the corresponding partition relation holds, then the corresponding selection hypothesis is true for the pair. What “appropriately related” means is defined in [12], and this plus the related results will be cited below.

Infinite games

The symbol $G_1(\mathcal{A}, \mathcal{B})$ denotes the game where two players, ONE and TWO, play an inning per positive integer. In the n th inning ONE chooses a set $O_n \in \mathcal{A}$, and TWO responds by choosing an element $T_n \in O_n$. TWO wins a play $(O_1, T_1, \dots, O_n, T_n, \dots)$ if $\{T_n : n \in \mathbb{N}\}$ is a member of \mathcal{B} ; otherwise, ONE wins. For the remainder of the paper, for a given topological space, \mathcal{O} denotes the collection of all open covers of the space. Galvin introduced the game $G_1(\mathcal{O}, \mathcal{O})$ in [5]. In [15] Pawlikowski proved that a space has property $S_1(\mathcal{O}, \mathcal{O})$ if, and only if, ONE has no winning strategy in the game $G_1(\mathcal{O}, \mathcal{O})$.

The symbol $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denotes the game where ONE and TWO play an inning per positive integer, and in the n th inning ONE chooses a set $O_n \in \mathcal{A}$, while TWO responds with a finite set $T_n \subseteq O_n$. TWO wins the play $(O_1, T_1, \dots, O_n, T_n, \dots)$ if $\bigcup_{n \in \mathbb{N}} T_n \in \mathcal{B}$; otherwise, ONE wins. Though Telgársky first explicitly defined a game of this form in [20], it was already considered in 1925 by Hurewicz when he proved in Theorem 10 of [9] that a space has property $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ if, and only if, ONE has no winning strategy in the game $G_{\text{fin}}(\mathcal{O}, \mathcal{O})$.

If ONE does not have a winning strategy in the game $G_1(\mathcal{A}, \mathcal{B})$ then the selection hypothesis $S_1(\mathcal{A}, \mathcal{B})$ is true; similarly for $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ and $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$. The converse implication—the selection hypothesis implies that ONE has no winning strategy in the corresponding game—is not always true, and accordingly of much greater interest. When this converse implication is true, the game characterizes the selection principle, and is a powerful tool to extract additional information about \mathcal{A} and \mathcal{B} . If \mathcal{A} has appropriate properties, the game-theoretic characterization can be used to derive Ramsey-theoretic statements. Such “appropriate” properties were identified in [12].

Part I: The selection principle $U_{\text{fin}}(\Gamma, \Omega)$

Let X be a space. An open cover \mathcal{U} of X is said to be a γ -cover if it is infinite, and each element of X belongs to all but finitely many elements of \mathcal{U} . Let Γ denote the collection of γ -covers of X . An open cover \mathcal{U} of X is said to be an ω -cover if X is not a member of \mathcal{U} , but for each finite subset F of X there is a $U \in \mathcal{U}$ such that $F \subseteq U$. Let Ω denote the collection of ω -covers of X .

The symbol $U_{\text{fin}}(\Gamma, \Omega)$ denotes the statement: For each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of γ -covers of X there is a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{V}_n \subset \mathcal{U}_n$, and either for some n we have $X = \bigcup \mathcal{V}_n$, or else $\{\bigcup \mathcal{V}_n: n \in \mathbb{N}\}$ is an ω -cover for X . In this section we shall show, in the spirit of [12], that this statement is equivalent to one of the form $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$, and can be characterized Ramsey-theoretically and game-theoretically.

Recall that an open cover \mathcal{U} of X is large if for each $x \in X$ the set $\{U \in \mathcal{U}: x \in U\}$ is infinite. The symbol Λ denotes the collection of large covers of X .

Definition 1. An open cover \mathcal{U} of X is *weakly groupable* if there is a partition $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ such that each \mathcal{U}_n is finite, for $m \neq n$ we have $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$, and for each finite subset F of X there is an n with $F \subset \bigcup \mathcal{U}_n$.

The symbol \mathcal{O}^{wgp} denotes the collection of all weakly groupable open covers of X . Similarly the symbol Λ^{wgp} denotes the collection of all weakly groupable large covers of X .

Theorem 2. For a space X the following are equivalent:

- (1) X has property $U_{\text{fin}}(\Gamma, \Omega)$.
- (2) X satisfies $S_{\text{fin}}(\Gamma, \Lambda^{wgp})$.
- (3) For each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of γ -covers of X there is a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of pairwise disjoint finite sets such that:
 - (a) for each n , $\mathcal{V}_n \subseteq \mathcal{U}_n$ and
 - (b) for each finite subset F of X there is an n with $F \subset \bigcup \mathcal{V}_n$.
- (4) X has property $S_{\text{fin}}(\Gamma, \Lambda)$, and each countable large cover of X is weakly groupable.
- (5) X has property $S_{\text{fin}}(\Omega, \Lambda)$, and each countable large cover of X is weakly groupable.
- (6) ONE has no winning strategy in the game $G_{\text{fin}}(\Gamma, \Lambda^{wgp})$.
- (7) ONE has no winning strategy in the game $G_{\text{fin}}(\Omega, \Lambda^{wgp})$.
- (8) X satisfies $S_{\text{fin}}(\Omega, \Lambda^{wgp})$.
- (9) X satisfies $S_{\text{fin}}(\Omega, \mathcal{O}^{wgp})$.
- (10) X satisfies $S_{\text{fin}}(\Gamma, \mathcal{O}^{wgp})$.

Proof. The implications (6) \Rightarrow (10), (7) \Rightarrow (8), (8) \Rightarrow (9) and (9) \Rightarrow (10) are easy. The rest deserve proof or further remarks.

(1) \Rightarrow (2): Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of γ -covers of X . We may assume that for $m \neq n$ we have $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$. We may also assume for each n that no finite subset of \mathcal{U}_n covers X .

Applying $U_{\text{fin}}(\Gamma, \Omega)$ we find for each n a finite $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\{\bigcup \mathcal{V}_n: n \in \mathbb{N}\}$ is an ω -cover of X . Thus $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a large cover of X and the partition $(\mathcal{V}_n: n \in \mathbb{N})$ witnesses that \mathcal{V} is weakly groupable.

(2) \Rightarrow (3): Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of γ -covers of X . We may assume that each \mathcal{U}_n is countable and that if $m \neq n$, then $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$. For each n enumerate \mathcal{U}_n bijectively as $(U_m^n: m \in \mathbb{N})$. Then for each n put

$$\mathcal{S}_n = \{U_k^1 \cap \dots \cap U_k^n: k \in \mathbb{N}\} \setminus \{\emptyset\}.$$

Then each \mathcal{S}_n is a γ -cover of X , and by omitting elements where necessary we may assume that for $m \neq n$ we have $\mathcal{S}_m \cap \mathcal{S}_n = \emptyset$. We also in advance choose for each element of each \mathcal{S}_n a representation as an intersection as in the definition.

Applying $\mathbf{S}_{\text{fin}}(\Gamma, \Lambda^{wsp})$ to the sequence $(\mathcal{S}_n: n \in \mathbb{N})$, we choose for each n a finite $\mathcal{W}_n \subset \mathcal{S}_n$ such that $\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{W}_n$ is in Λ^{wsp} , and for $m \neq n$ also $\mathcal{W}_m \cap \mathcal{W}_n = \emptyset$. Next, write

$$\mathcal{W} = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n$$

where for each n \mathcal{R}_n is finite, and for $m \neq n$ also $\mathcal{R}_m \cap \mathcal{R}_n = \emptyset$, and for each finite $F \subset X$ there is an m with $F \subset \bigcup \mathcal{R}_m$.

Choose $i_1 \geq 1$ minimal such that for all $j > i_1$ we have $\mathcal{W}_1 \cap \mathcal{R}_j = \emptyset$. Then put

$$\mathcal{V}_1 = \left\{ U_k^1: (\exists j \leq i_1) (U_k^1 \text{ a term in the chosen representation of an element of } \mathcal{R}_j) \right\}.$$

Then choose $i_2 > i_1$ minimal so that for all $j > i_2$ we have $\mathcal{W}_2 \cap \mathcal{R}_j = \emptyset$. Put

$$\mathcal{V}_2 = \left\{ U_k^2: (\exists j \leq i_2) (U_k^2 \text{ a term in the chosen representation of an element of } \mathcal{R}_j) \right\}.$$

In general, choose $i_{k+1} > i_k$ minimal such that for all $j > i_{k+1}$ we have $\mathcal{W}_{k+1} \cap \mathcal{R}_j = \emptyset$ and put

$$\mathcal{V}_{k+1} = \left\{ U_\ell^{k+1}: (\exists j \leq i_{k+1}) (U_\ell^{k+1} \text{ a term in the chosen representation of an element of } \mathcal{R}_j) \right\}.$$

Then for each k we have $\mathcal{V}_k \subset \mathcal{U}_k$ finite, and for $m \neq n$ we have $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$. Let $F \subseteq X$ be a finite set. Find an n with $F \subseteq \bigcup \mathcal{R}_n$. Find the least k with $n \leq i_k$. Then we have $i_{k-1} < n$ and so $(\mathcal{W}_1 \cup \dots \cup \mathcal{W}_{k-1}) \cap \mathcal{R}_n = \emptyset$. But then each element of \mathcal{R}_n has in its representation a term of the form U_i^k . It follows that $\bigcup \mathcal{R}_n \subseteq \bigcup \mathcal{V}_k$, and so $F \subseteq \bigcup \mathcal{V}_k$.

(3) \Rightarrow (4): It is evident that $\mathbf{S}_{\text{fin}}(\Gamma, \Lambda^{wsp})$ implies $\mathbf{S}_{\text{fin}}(\Gamma, \Lambda)$. We must show that $\mathbf{S}_{\text{fin}}(\Gamma, \Lambda^{wsp})$ implies that every countable large cover of X is in Λ^{wsp} . To this end let $(U_n: n \in \mathbb{N})$ bijectively enumerate a countable large cover \mathcal{U} of X . We may assume that no finite subset of \mathcal{U} covers X .

For each m define

$$\mathcal{V}_m = \left\{ \bigcup_{m < j \leq n} U_j: n \in \mathbb{N} \right\}.$$

Then each \mathcal{V}_m is a γ -cover of X . We may assume that for $m \neq n$ we have $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$. Apply 3 to $(\mathcal{V}_n: n \in \mathbb{N})$, and choose for each n a finite set $\mathcal{W}_n \subset \mathcal{V}_n$ such that for $m \neq n$ we have $\mathcal{W}_m \cap \mathcal{W}_n = \emptyset$, and for each finite set $F \subset X$ there is an n such that $F \subset \bigcup \mathcal{W}_n$.

Put $k_0 = m_0 = n_0 = 1$, and proceed as follows:

Choose $m_1 = 2$. Then $\mathcal{W}_1 \subseteq \bigcup_{j \leq m_1} \mathcal{W}_j$.

Choose $n_1 \geq m_1$ minimal such that for each element of $\bigcup_{j \leq m_1} \mathcal{W}_j$, if U_i is a term in the chosen representation of an element of it then $i < n_1$.

Choose $k_1 > n_1$ so large that for all $j \geq k_1$ we have:

- (1) If U_i is a term in the chosen representation of an element of \mathcal{W}_j , then $i \geq n_1$;
- (2) k_1 is minimal subject to 1 and $k_1 > n_1$.

Next, choose $m_2 = k_1 + 1$. Choose $n_2 \geq m_2$ minimal such that for each element of $\bigcup_{j \leq m_2} \mathcal{W}_j$, if U_i is a term in its chosen representation, then $i < n_2$. Choose $k_2 > n_2$ such that for all $j \geq k_2$ we have:

- (1) If U_i is a term in the chosen representation of an element of \mathcal{W}_j , then $i \geq n_2$;
- (2) k_2 is minimal subject to 1 and $k_2 > n_2$.

In general, choose $m_{j+1} > k_j + 1$. Choose $n_{j+1} \geq m_{j+1}$ minimal such that for each element $\bigcup_{i \leq m_{j+1}} \mathcal{W}_i$, if U_i is a term in its chosen representation, then $t \leq n_{j+1}$;

Choose $k_{j+1} \geq n_{j+1}$ minimal so that if $t \geq k_{j+1}$ then:

- (1) If U_i is a term in the chosen representation of an element of \mathcal{W}_t , then $i \geq n_{j+1}$;
- (2) k_{j+1} is minimal subject to 1 and $k_{j+1} > n_{j+1}$.

For each n put $B_n = \bigcup_{k_{n-1}+1 \leq j \leq k_n} \mathcal{W}_j$. One can check that for each m it is the case that

$$B_m \subseteq \bigcup_{n_{m-1} \leq i < n_{m+1}} U_i.$$

But by the choice of the \mathcal{W}_i 's we have: Either for each finite $F \subset X$ there is an n such that $F \subset B_{2 \cdot n - 1}$, or else for each finite $F \subset X$ there is an n such that $F \subset B_{2 \cdot n}$.

In the former case the partition

$$\{U_i: n_0 \leq i < n_2\}, \{U_i: n_2 \leq i < n_4\}, \dots, \{U_i: n_{2 \cdot m} \leq i < n_{2 \cdot m + 2}\}, \dots$$

of the large cover \mathcal{U} witnesses that \mathcal{U} is in Λ^{wgp} . In the latter case the partition

$$\{U_i: n_1 \leq i < n_3\}, \{U_i: n_3 \leq i < n_5\}, \dots, \{U_i: n_{2 \cdot m - 1} \leq i < n_{2 \cdot m + 1}\}, \dots$$

witnesses the same. This completes the proof that (3) implies (4).

(4) \Rightarrow (5): It was shown in [10] that $\mathbf{S}_{\text{fin}}(\Gamma, \Lambda)$ is equivalent to $\mathbf{S}_{\text{fin}}(\Omega, \Lambda)$. This plus (4) gives (5).

(4) \Rightarrow (6): By (4) and Corollary 5 and Proposition 11 of [17] and Theorem 5 of [19] ONE has no winning strategy in the game $\mathbf{G}_{\text{fin}}(\Lambda, \Lambda)$. Since a strategy for ONE in $\mathbf{G}_{\text{fin}}(\Gamma, \Lambda)$ is also a strategy for ONE in $\mathbf{G}_{\text{fin}}(\Lambda, \Lambda)$, ONE has no winning strategy in $\mathbf{G}_{\text{fin}}(\Gamma, \Lambda)$. In any play lost by ONE in the game $\mathbf{G}_{\text{fin}}(\Gamma, \Lambda)$, player TWO ends up with a countable large cover of X . By (4) this cover is indeed weakly groupable. Thus, ONE has no winning strategy in $\mathbf{G}_{\text{fin}}(\Gamma, \Lambda^{wgp})$.

(5) \Rightarrow (7): By the methods of [17] or [10] $\mathbf{S}_{\text{fin}}(\Omega, \Lambda)$ is equivalent to $\mathbf{S}_{\text{fin}}(\Lambda, \Lambda)$. Then as in the proof of (4) \Rightarrow (6) one concludes that ONE has no winning strategy in $\mathbf{G}_{\text{fin}}(\Omega, \Lambda^{wgp})$.

(10) \Rightarrow (3): This is proved similarly to proving that (2) \Rightarrow (3).

(3) \Rightarrow (1): This is easy. \square

Theorem 3. For a space X the following are equivalent:

- (1) $\mathbf{S}_{\text{fin}}(\Omega, \Lambda^{wgp})$;
- (2) For each k the partition relation $\Omega \rightarrow [\Lambda^{wgp}]_k^2$ holds.

Proof. (1) \Rightarrow (2): $\mathbf{S}_{\text{fin}}(\Omega, \Lambda^{wgp})$ implies that ONE has no winning strategy in the game $\mathbf{G}_{\text{fin}}(\Omega, \Lambda^{wgp})$. By [12] Theorem 2: For each k , $\Omega \rightarrow [\Lambda^{wgp}]_k^2$.

(3) \Rightarrow (1): By [12], Theorem 5: $\Omega \rightarrow [\Lambda^{wgp}]_k^2$ implies $\mathbf{S}_{\text{fin}}(\Omega, \Lambda^{wgp})$. \square

The results of this section and one of the main results of [12] illuminate somewhat the currently still open Problems 1 and 2 on p. 243 of [10]. For recall from [12] that an open cover \mathcal{U} of a space X is *groupable* if there is a partition $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ such that each \mathcal{U}_n is finite, for $m \neq n$ we have $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$, and for each $x \in X$, for all but finitely many n we have $x \in \bigcup \mathcal{U}_n$. The symbols \mathcal{O}^{gp} and Λ^{gp} denote the families of groupable open covers, and groupable large covers of X , respectively.

Problem 1 of [10] translates to the problem

Problem 1. Is $\mathbf{S}_{\text{fin}}(\Gamma, \Lambda^{wgp}) = \mathbf{S}_{\text{fin}}(\Gamma, \Omega)$?

Problem 2 of [10] translates to the problem:

Problem 2. And if not, does $\mathbf{S}_{\text{fin}}(\Gamma, \Lambda^{gp})$ imply $\mathbf{S}_{\text{fin}}(\Gamma, \Omega)$?

The selection principle $\mathbf{S}_1(\Omega, \Lambda^{wgp})$

We now discuss the stronger selection principle $\mathbf{S}_1(\Omega, \Lambda^{wgp})$. As before the key observation that brings techniques developed elsewhere to bear on this selection principle is:

Lemma 4. The following statements are equivalent:

- (1) $\mathbf{S}_1(\Omega, \Lambda^{wgp})$.
- (2) $\mathbf{S}_1(\Omega, \Lambda)$ and each countable large cover is weakly groupable.

Proof. Only (1) \Rightarrow (2) requires proof. It is evident that $\mathbf{S}_1(\Omega, \Lambda^{wgp})$ implies each of $\mathbf{S}_1(\Omega, \Lambda)$, and $\mathbf{S}_{\text{fin}}(\Omega, \Lambda^{wgp})$. Apply (5) of Theorem 2. \square

This is used in the proof of the main result of this subsection:

Theorem 5. For a space X the following are equivalent:

- (1) $\mathbf{S}_1(\Omega, \Lambda^{wgp})$.
- (2) ONE has no winning strategy in the game $\mathbf{G}_1(\Omega, \Lambda^{wgp})$.
- (3) For each k , $\Omega \rightarrow (\Lambda^{wgp})_k^2$.

Proof. (1) \Rightarrow (2): Let F be a strategy of ONE in the game $G_1(\Omega, \Lambda^{wgp})$. Then it is also a strategy for ONE in the game $G_1(\Omega, \Lambda)$, and thus a strategy for ONE in the game $G_1(\Lambda, \Lambda)$. But X has property $S_1(\Omega, \Lambda)$. Then by Theorem 17 of [17] X has the property $S_1(\Lambda, \Lambda)$. By Theorem 3 of [19], F is not a winning strategy for ONE in the game $G_1(\Lambda, \Lambda)$ —and thus not in the game $G_1(\Omega, \Lambda)$. Consider any F -play of this game, lost by ONE, say

$$F(\emptyset), T_1, F(T_1), T_2, F(T_1, T_2), T_3, \dots$$

Then $\{T_n: n \in \mathbb{N}\}$ is a countable large cover of X . Again using (1) and Lemma 4 we see that this large cover of X is weakly groupable, and thus this play is indeed a play of $G_1(\Omega, \Lambda^{wgp})$ that is lost by ONE.

(2) \Rightarrow (3): This implication follows directly from [12], Theorem 1.

(3) \Rightarrow (1): Theorem 6 of [12] gives this implication. \square

Problem 3. Is $S_1(\Omega, \Lambda^{wgp})$ stronger than $S_1(\Omega, \Lambda)$?

Problem 4. Is $S_1(\Omega, \Omega)$ stronger than $S_1(\Omega, \Lambda^{wgp})$?

Part II: The relative Hurewicz property

In this part of the paper we give a detailed development of what we call the relative Hurewicz property. Some of the work here improves and extends results from [8].

W. Hurewicz introduced the Hurewicz covering property in 1925 in [9]:

Definition 6. For each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of open covers of a space X there is a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of finite sets such that for each n $\mathcal{V}_n \subset \mathcal{U}_n$, and for each $x \in X$, for all but finitely many n , $x \in \bigcup \mathcal{V}_n$.

We shall refer to the property described in Definition 6 as the *absolute* Hurewicz property. The relative version of this property is:

Definition 7 [8]. Let Y be a subset of a space X . We say that Y is *Hurewicz in X* (or *relatively Hurewicz in X*) if for each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ such that every \mathcal{V}_n is a finite subset of \mathcal{U}_n and each $y \in Y$ belongs to $\bigcup \mathcal{V}_n$ for all but finitely many n .

Neither of these two definitions define a selection property of the form $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$. But in [12] it has been shown that for appropriately chosen classes \mathcal{A} and \mathcal{B} of open covers of a space X the absolute Hurewicz property can be characterized by a selection principle of the form $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$. Our first task in this part is to also characterize the relative Hurewicz property in this way.

One of the important tools in executing this task is the following game-theoretic characterization of the relative Hurewicz property. With Y a subspace of the space X , let the game $H(Y, X)$ be as follows: Players ONE and TWO play an inning per positive integer.

In the n th inning ONE first chooses an open cover O_n of X , and then TWO responds with a finite subset T_n of O_n . A play

$$O_1, T_1, O_2, T_2, \dots, O_n, T_n, \dots$$

is won by TWO if for each $y \in Y$ and for all but finitely many n , we have $y \in \bigcup T_n$.

Theorem 8 [1]. *For a subspace Y of a Lindelöf space X the following are equivalent:*

- (1) Y is relative Hurewicz in X .
- (2) ONE has no winning strategy in the game $H(Y, X)$.

Theorem 9. *For a Lindelöf space X and its subspace Y the following are equivalent:*

- (1) Y has the Hurewicz property in X ;
- (2) $\mathfrak{S}_{\text{fin}}(\Lambda_X, \Lambda_Y^{\text{gp}})$ holds;
- (3) $\mathfrak{S}_{\text{fin}}(\Omega_X, \Lambda_Y^{\text{gp}})$ holds.

Proof. (1) \Rightarrow (2): Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a given sequence of large covers of X . One may assume that these covers are countable.

Consider the following strategy, σ , of ONE in the game $H(Y, X)$. The first move by ONE is $\sigma(\emptyset) = \mathcal{U}_1$. If TWO responds with the finite set $T_1 \subset \sigma(\emptyset)$, then ONE plays $\sigma(T_1) = \mathcal{U}_2 \setminus T_1$, still a large cover of X . Suppose n innings have been played and TWO's responses in these were the finite sets T_1, \dots, T_n . Then ONE's response is $\sigma(T_1, \dots, T_n) = \mathcal{U}_{n+1} \setminus (T_1 \cup \dots \cup T_n)$. This defines a legitimate strategy for ONE.

Apply the fact that Y is Hurewicz in X : By Theorem 8 σ is not a winning strategy for ONE. Consider a play

$$\sigma(\emptyset), T_1, \sigma(T_1), T_2, \sigma(T_1, T_2), \dots$$

which is lost by ONE.

Then for each $y \in Y$, for all but finitely many n we have $y \in \bigcup T_n$. Also, by the definition of σ , the finite sets T_n are disjoint from each other. But then $\bigcup_{n \in \mathbb{N}} T_n$ is a groupable large cover of Y .

(2) \Rightarrow (3): This follows directly from the fact that every ω -cover is large.

(3) \Rightarrow (1): Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of open covers of X . We may assume that each \mathcal{U}_n is countable and does not contain a finite cover of X .

For each n , let \mathcal{V}_n be the set of finite unions of elements of \mathcal{U}_n . Then each \mathcal{V}_n is an ω -cover of X . Enumerate each \mathcal{V}_n bijectively as $(V_k^n: k \in \mathbb{N})$. Define new ω -covers \mathcal{W}_n in the following way:

- (1) $\mathcal{W}_1 = \mathcal{V}_1$;
- (2) For $n > 1$, $\mathcal{W}_n = \{V_{m_1}^1 \cap V_{m_2}^2 \cap \dots \cap V_{m_n}^n : n < m_1 < m_2 < \dots < m_n\} \setminus \{\emptyset\}$.

For each element of \mathcal{W}_n choose a representation of the form $V_{m_1}^1 \cap V_{m_2}^2 \cap \dots \cap V_{m_n}^n$ with $n < m_1 < m_2 < \dots < m_n$.

Apply $S_{\text{fin}}(\Omega_X, \Lambda_Y^{gp})$ to the sequence $(\mathcal{W}_n: n \in \mathbb{N})$ to find for each n a finite set $\mathcal{G}_n \subset \mathcal{W}_n$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ is a groupable large cover of Y . Choose for each n a finite set \mathcal{H}_n such that these are disjoint from each other, $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$, and each element of Y belongs to all but finitely many of the sets $\bigcup \mathcal{H}_n$.

Choose $n_1 > 1$ so large that $\mathcal{H}_{n_1} \subset \bigcup_{j>1} \mathcal{G}_j$. Then let \mathcal{Z}_1 be the set of V_k^1 that appear as terms in the chosen representations of elements of \mathcal{H}_{n_1} . Then choose $n_2 > n_1$ so large that $\mathcal{H}_{n_2} \subset \bigcup_{j>2} \mathcal{G}_j$ and let \mathcal{Z}_2 be the set of V_k^2 that appear as terms in the chosen representations of elements of \mathcal{H}_{n_2} , and so on. In this way we obtain finite sets $\mathcal{Z}_n \subset \mathcal{V}_n$ such that each element of Y belongs to all but finitely many of the sets $\bigcup \mathcal{Z}_n$.

Finally, for each element C of \mathcal{Z}_n choose finitely many elements of \mathcal{U}_n whose union produces C and let \mathcal{C}_n denote the finite set of elements of \mathcal{U}_n chosen in this way. Then the sequence $(\mathcal{C}_n: n \in \mathbb{N})$ witnesses the relative Hurewicz property of Y in X for the given sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of open covers. \square

The next two theorems give a characterization of the relative Hurewicz property in all finite powers. According to [7] a space is said to be an ε -space if each ω -cover contains a countable subset which still is an ω -cover. Arkhangel'skiĭ and Pytkeev showed that this is equivalent to having the Lindelöf property in all finite powers.

Theorem 10. *For an ε -space X and a subspace Y of X : If for each $n \in \mathbb{N}$, Y^n is Hurewicz in X^n then $S_{\text{fin}}(\Omega_X, \Omega_Y^{gp})$ holds.*

Proof. Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of ω -covers of X . We may assume that each is countable. If for each n Y^n has the Hurewicz property in X^n , then $\mathcal{Y} = \sum_{n \in \mathbb{N}} Y^n$ has the Hurewicz property in $\mathcal{X} = \sum_{n \in \mathbb{N}} X^n$. Then by Theorem 8 ONE has no winning strategy in the game $H(\mathcal{X}, \mathcal{Y})$. Also, for each n the set $\mathcal{O}_n = \{U^m: U \in \mathcal{U}_n, m \in \mathbb{N}\}$ is a large open cover of \mathcal{X} .

Consider the strategy σ of ONE defined as follows: In the first inning, $\sigma(\emptyset) = \mathcal{O}_1$. If TWO responds with the finite set $T_1 \subset \mathcal{O}_1$, then ONE plays $\sigma(T_1) = \mathcal{O}_2 \setminus \{V^n: n \in \mathbb{N} \text{ and } (\exists j)(V^j \in T_1)\}$. Supposing now that during the first n innings TWO has played the finite sets T_1, \dots, T_n , ONE plays in the next inning the set $\sigma(T_1, \dots, T_n) = \mathcal{O}_{n+1} \setminus \{V^n: n \in \mathbb{N} \text{ and } (\exists j)(V^j \in (T_1 \cup \dots \cup T_n))\}$. This defines σ for all legitimate moves in the game.

By Theorem 8, σ is not a winning strategy for ONE. Consider a play

$$\sigma(\emptyset), T_1, \sigma(T_1), T_2, \sigma(T_1, T_2), \dots$$

which is lost by ONE. Then, for each $y \in \mathcal{Y}$, for all but finitely many n , we have $y \in \bigcup T_n$. For each n , define $\mathcal{V}_n = \{V: (\exists j)(V^j \in T_n)\}$. Each \mathcal{V}_n is finite. By the definition of σ we have $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$ whenever $m \neq n$. Moreover, if F is a finite subset of Y then for all but finitely many n there is a $V \in \mathcal{V}_n$ such that $F \subset V$. Finally, for each n we have $\mathcal{V}_n \subset \mathcal{U}_n$. It follows that $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a groupable ω -cover of Y . \square

Let \mathcal{A} and \mathcal{B} be families of subsets of the infinite set S . Then $\text{CDR}_{\text{sub}}(\mathcal{A}, \mathcal{B})$ denotes the statement that for each sequence $(A_n: n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(B_n: n \in \mathbb{N})$ such that for each n $B_n \subseteq A_n$, for $m \neq n$, $B_m \cap B_n = \emptyset$, and each B_n is a member of \mathcal{B} . This notation was introduced in [17].

Lemma 11. *Let X be an ε -space for which $\text{CDR}_{\text{sub}}(\Omega, \Omega)$ holds. If for a subspace Y of X $\text{S}_{\text{fin}}(\Omega_X, \Omega_Y^{\text{sp}})$ holds, then for each sequence $(\mathcal{U}_n: n \in \mathbb{N})$ of ω -covers of X there is a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ of finite sets such that*

- (1) $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$ whenever $m \neq n$;
- (2) For each n , $\mathcal{V}_n \subset \mathcal{U}_n$;
- (3) For each finite $F \subset Y$, for all but finitely many n there is a $V \in \mathcal{V}_n$ such that $F \subset V$.

Proof. Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of ω -covers of X . Since X has the property $\text{CDR}_{\text{sub}}(\Omega, \Omega)$ we may assume that the \mathcal{U}_n 's are disjoint from each other. Since X is an ε -space we may also assume that each \mathcal{U}_n is countable. Enumerate each \mathcal{U}_n bijectively as $(U_k^n: k \in \mathbb{N})$.

For each n define \mathcal{V}_n to be the collection of nonempty sets of the form

$$U_{m_1}^1 \cap \dots \cap U_{m_n}^n.$$

Then each \mathcal{V}_n is an ω -cover of X . Apply $\text{S}_{\text{fin}}(\Omega_X, \Omega_Y^{\text{sp}})$ to this sequence to find for each n a finite nonempty set $\mathcal{V}'_n \subset \mathcal{V}_n$ such that $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}'_n$ is a groupable ω -cover of Y .

Select a partition $\mathcal{W} = \bigcup_{k \in \mathbb{N}} \mathcal{W}_k$ of \mathcal{V} such that each \mathcal{W}_k is finite, and for each finite subset F of Y , for all but finitely many k there is a $W \in \mathcal{W}_k$ such that $F \subset W$. Put $k_1 = 1$ and let \mathcal{H}_1 be the set of U_i^1 's that occur as terms in the chosen representations of elements of \mathcal{W}_{k_1} . Next choose $k_2 > k_1$ so large that for all $j \geq k_2$ we have \mathcal{W}_j disjoint from \mathcal{V}'_1 . Let \mathcal{H}_2 consist of all sets of the form U_i^2 that occur as a term in the chosen representation of an element of \mathcal{W}_{k_2} . Then choose $k_3 > k_2$ so large that for all $j \geq k_3$ we have $\mathcal{W}_j \cap \mathcal{V}'_2 = \emptyset$, and let \mathcal{H}_3 consist of the U_i^3 's that occur as terms in the chosen representations of elements of \mathcal{W}_{k_3} , and so on.

In this way we obtain a sequence $(\mathcal{H}_n: n \in \mathbb{N})$ of finite sets as required. \square

Theorem 12. *Let X be an ε -space for which $\text{CDR}_{\text{sub}}(\Omega, \Omega)$ holds. If for a subspace Y of X $\text{S}_{\text{fin}}(\Omega_X, \Omega_Y^{\text{sp}})$ holds, then for each $n \in \mathbb{N}$, Y^n is Hurewicz in X^n .*

Proof. Fix n and let $(\mathcal{W}_k: k \in \mathbb{N})$ be a sequence of large covers of X^n . For each k let \mathcal{U}_k be the collection of open subsets V of X such that V^n is a subset of a union of finitely many elements of \mathcal{W}_k . Then each \mathcal{U}_k is an ω -cover of X : For let a finite subset F of X be given. Let $\mathcal{W}_{F,k}$ be a finite subset of \mathcal{W}_k which covers F^n . Since F^n is compact Wallace's theorem implies that there is an open set $V \subset X$ such that $F^n \subset V^n \subset \bigcup \mathcal{W}_{F,k}$. Thus, $F \subset V$ and $V \in \mathcal{U}_k$.

Apply $\text{S}_{\text{fin}}(\Omega_X, \Omega_Y^{\text{sp}})$ and Lemma 11 and choose for each k a finite set $\mathcal{V}_k \subset \mathcal{U}_k$ such that the sequence $(\mathcal{V}_k)_{k \in \mathbb{N}}$ has the three properties of Lemma 11. For each k and for each element V of \mathcal{V}_k choose finitely many elements of \mathcal{W}_k which cover V^n ; let \mathcal{H}_k be the finite set of elements of \mathcal{W}_k chosen in this way. Then the sequence $(\mathcal{H}_k: k \in \mathbb{N})$ witnesses for $(\mathcal{W}_k: k \in \mathbb{N})$ that Y^n is Hurewicz in X^n . \square

Part III: The Hurewicz property in metrizable spaces

In this part we treat for the Hurewicz covering property a topic which is more extensively studied in [1] and [2] for several selection principles. Like for the Rothberger property, there is a useful way of describing in metrizable spaces those subspaces which are relatively Hurewicz in the superspace in terms of basis properties of the superspace and in terms of metric measure theory. The analogous work for the Rothberger property $S_1(\mathcal{O}, \mathcal{O})$ and its relative version appears in part in [14] and in part in [18].

In [13] Menger defined the following: Metric space (X, d) has the *Menger basis property* if there is for each basis \mathcal{B} of (X, d) a sequence $(U_n: n \in \mathbb{N})$ such that $\lim_{n \rightarrow \infty} \text{diam}_d(U_n) = 0$, and $\{U_n: n \in \mathbb{N}\}$ covers X . In [9] Hurewicz showed that the Menger basis property is equivalent to the Menger covering property $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$. When the spaces in question are metrizable the relative version of the Menger property can be similarly characterized by a relativized Menger basis property.

We show now that also the Hurewicz property and its relative version are characterized by a basis property. Let Y be a subset of a metrizable space X .

Definition 13. Y has the *Hurewicz basis property* in X if for any basis \mathcal{B} of metric space (X, d) there is a sequence $(U_n: n \in \mathbb{N})$ from \mathcal{B} such that $\{U_n: n \in \mathbb{N}\}$ is a groupable cover for Y and $\lim_{n \rightarrow \infty} \text{diam}_d(U_n) = 0$.

Theorem 14. Let (X, d) be a metric space with no isolated points and let Y be a subspace of X . The following statements are equivalent:

- (1) Y is relatively Hurewicz in X ;
- (2) Y has the Hurewicz basis property in X .

Proof. (1) \Rightarrow (2): Let Y be relatively Hurewicz in X . Let \mathcal{B} be a basis of X and let $\mathcal{U}_n = \{U \in \mathcal{B} \mid \text{diam}_d(U) < \frac{1}{n+1}\}$. Each \mathcal{U}_n is a large open cover of X . Then for each n let $\mathcal{V}_n \subseteq \mathcal{U}_n$ be a finite set such that $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a groupable cover for Y . If $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n = \{U_n: n \in \mathbb{N}\}$, then $\lim_{n \rightarrow \infty} \text{diam}_d(U_n) = 0$.

(1) \Leftarrow (2): Let Y have the Hurewicz basis property in X and let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of open covers of X . We may assume without loss of generality that whenever an open set V is a subset of an element of a \mathcal{U}_n , then V is a member of \mathcal{U}_n . For each n define $\mathcal{H}_n = \{U_1 \cap \dots \cap U_n: (\forall i \leq n)(U_i \in \mathcal{U}_i)\} \setminus \{\emptyset\}$. Then still each \mathcal{H}_n is an open cover of X , and has the property that whenever an open set V is a subset of a member of \mathcal{H}_n then $V \in \mathcal{H}_n$.

Let \mathcal{U} be the set $\{U \cup V \mid (\exists n)(U, V \in \mathcal{H}_n \text{ and } \text{diam}_d(U \cup V) > \frac{1}{n})\}$.

Claim. \mathcal{U} is a basis for X .

For let W be an open subset of X containing a point $x \in W$. Since x is not an isolated point of X , choose $y \in W \setminus \{x\}$ and $n > 1$ with $d(x, y) > \frac{1}{n}$. Then find $U, V \in \mathcal{H}_n$ with $x \in U$, $y \in V$ and $U \cup V \subseteq W$. This completes the proof of the claim.

Since Y has the Hurewicz basis property in X select sets W_n from the basis \mathcal{U} of X such that $\lim_{n \rightarrow \infty} \text{diam}_d(W_n) = 0$ and $\{W_n: n \in \mathbb{N}\}$ is a groupable cover for Y .

Select a sequence $m_1 < m_2 < \dots < m_k < \dots$ such that for each $y \in Y$, for all but finitely many k there is a j with $m_k \leq j < m_{k+1}$ such that $y \in W_j$.

For each n choose the least k_n and sets U_n and V_n from \mathcal{U}_{k_n} such that $W_n = U_n \cup V_n$, and choose m_n maximal with $\text{diam}_d(W_n) < \frac{1}{m_n}$. Then $k_n > m_n$ for each n , and $\lim_{n \rightarrow \infty} m_n = \infty$. Hence for each such selected k_n there are only finitely many W_n for which the chosen representatives U_n, V_n are from \mathcal{U}_{k_n} and have $\text{diam}_d(U_n \cup V_n) > \frac{1}{k_n}$. Let \mathcal{V}_{k_n} be the finite set of such U_n, V_n . For convenience let us also say that W_n uses \mathcal{U}_{k_n} .

Now choose $\ell_1 < \ell_2 < \dots < \ell_m < \dots$ and $j_1 < j_2 < \dots < j_m < \dots$ as follows:

Choose $\ell_1 > 1$ so large that each W_i with $i \leq m_1$ has a representation of the form $U \cup V$ using U 's and V 's from the sets \mathcal{V}_{k_i} , $k_i \leq \ell_1$. Then choose j_1 so large that for all $i > j_1$, if W_i uses a \mathcal{V}_{k_n} , then $k_n > \ell_1$.

To define ℓ_2 , let m_k be least larger than j_1 , and now choose ℓ_2 so large that if W_i with $m_k < i < m_{k+1}$ uses a \mathcal{U}_{k_i} then $k_i \leq \ell_2$. Then choose $j_2 > j_1$ so large that for all $i \geq j_2$, if W_i uses a \mathcal{V}_{k_n} then $k_n > \ell_i$.

Continue in this way to alternately choose ℓ_m and j_m . Observe for each m that if we consider the least $m_k > \ell_m$, then:

- (1) if W_i with $m_k \leq i < m_{k+1}$ uses a \mathcal{V}_{k_i} then $\ell_m < k_i \leq \ell_{m+1}$;
- (2) if $i \geq j_m$, then if W_i uses \mathcal{V}_{k_n} , then $k_n > \ell_m$.

For each $V \in \mathcal{U}_{k_n}$ with $k_n \leq \ell_1$ choose a $U \in \mathcal{U}_1$ with $V \subseteq U$, and let $\mathcal{G}_1 \subseteq \mathcal{U}_1$ be this finite set.

In general for each $V \in \mathcal{U}_{k_n}$ with $\ell_p < k_n \leq \ell_{p+1}$ choose a $U \in \mathcal{U}_p$ with $V \subseteq U$. Let $\mathcal{G}_p \subseteq \mathcal{U}_p$ be this finite set.

Then we have that for each $y \in Y$, for all but finitely many p , $y \in \bigcup \mathcal{G}_p$. It follows that Y is Hurewicz in X . \square

In Theorem 14 it is necessary to assume that X has no isolated points. For example, suppose $X = \mathbb{N}$ and has the discrete topology. Let Y be any infinite subset of X and let \mathcal{B} be the basis $\{\{x\}: x \in X\}$ of X . Then no sequence from \mathcal{B} is a groupable cover of Y .

In [4] Borel defined a notion nowadays called *strong measure zero*. In light of new developments in the combinatorics of open covers (see [1] and [2]) it seems more appropriate to call Borel's property *Borel strong measure zero*: Y is *Borel strong measure zero* if there is for each sequence $(\varepsilon_n: n \in \mathbb{N})$ of positive real numbers a sequence $(J_n: n \in \mathbb{N})$ of subsets of Y such that each J_n is of diameter $< \varepsilon_n$, and Y is covered by $\{J_n: n \in \mathbb{N}\}$.

In [18] it was shown that if Y is a subset of a σ -compact metrizable space X then Y has the relative Rothberger property in X if, and only if, Y has Borel strong measure zero with respect to each metric on X which generates the topology of X .

In [14] it was shown that Y has the absolute Rothberger property if, and only if, it has Borel strong measure zero with respect to each metric on Y which generates the topology of Y .

We now give a similar description in terms of metrization theory of the relative- and absolute-Hurewicz properties. First we define:

Definition 15. Metric space (Y, d) is *Hurewicz measure zero* if there is for each sequence $(\varepsilon_n: n \in \mathbb{N})$ of positive real numbers a sequence $(\mathcal{V}_n: n \in \mathbb{N})$ such that:

- (1) for each n , \mathcal{V}_n is a finite set of subsets of Y ;
- (2) for each n , each member of \mathcal{V}_n has d -diameter less than ε_n ;
- (3) $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a groupable cover of Y .

Theorem 16. Let (X, d) be a zero-dimensional separable metric space with no isolated points and let Y be a subspace of X . The following statements are equivalent:

- (a) Y has the Hurewicz property in X ;
- (b) Y has Hurewicz measure zero with respect to every metric on X which gives X the same topology as d does.

Proof. (a) \Rightarrow (b): Let Y have the relative Hurewicz property in X , let d be any metric on X and let $(\varepsilon: n \in \mathbb{N})$ be any sequence of positive real numbers. For each $n \in \mathbb{N}$ define $\mathcal{U}_n = \{U \subseteq X \mid U \text{ is an open set of } d\text{-diameter at most } \varepsilon_n\}$. Then each \mathcal{U}_n is a large open cover of X . Since Y has the Hurewicz property in X there exist by Theorem 9 finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a groupable cover of Y . Hence Y has Hurewicz measure zero with respect to every metric d of X which gives X the same topology.

(b) \Rightarrow (a): Let d be an arbitrary metric on X which gives X the same topology as the original one. Let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of open covers of X . Since X is a separable zero-dimensional metric space, we can find $(\mathcal{U}_n^*: n \in \mathbb{N})$ such that for each n :

- (1) \mathcal{U}_n^* is clopen disjoint cover of Y refining \mathcal{U}_n ;
- (2) $U \in \mathcal{U}_n^*$ implies that $\text{diam}_d(U) \leq \frac{1}{n}$;
- (3) \mathcal{U}_{n+1}^* refines \mathcal{U}_n^* .

(To see that this can be done: First replace the cover \mathcal{U}_n by the cover $\{U \mid U \text{ clopen, } \text{diam}_d(U) < \frac{1}{n} \text{ and } \exists V \in \mathcal{U}_n, U \subseteq V\}$. Since X is separable we can replace this last cover by a countable subcover $\{U_m: m \in \mathbb{N}\}$. Since the sets in this cover are clopen we can make the cover disjoint. Finally obtain (3) by further intersections.)

Now define a metric d^* on X by $d^*(x, y) = \frac{1}{n+1}$ where n is the least such that there exist $U \in \mathcal{U}_n^*$ with $x \in U$ and $y \notin U$. One can check that d^* generates the same topology on X as d does. Thus Y has Hurewicz measure zero with respect to d^* . By setting $\varepsilon_n = \frac{1}{n+1}$ for each n , we find finite sets \mathcal{V}_n such that $\text{diam}_{d^*}(U)$ is less than $\varepsilon_n (= \frac{1}{n+1})$ whenever $U \in \mathcal{V}_n$, and $\bigcup \{\mathcal{V}_n: n \in \mathbb{N}\}$ is groupable cover for Y .

Let $(\mathcal{W}_n: n \in \mathbb{N})$ be a sequence of finite families of open sets such that $\mathcal{W}_m \cap \mathcal{W}_n = \emptyset$ whenever $m \neq n$, and $\bigcup_{n \in \mathbb{N}} \mathcal{W}_n = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$, and for each $y \in Y$, for all but finitely many n , $y \in \bigcup \mathcal{W}_n$.

Choose sequences $1 < i_1 < i_2 < \dots < i_m < \dots$ and $j_0 = 1 < j_1 < j_2 < \dots < j_m < \dots$ such that:

- (1) Each element of \mathcal{W}_1 is contained in $\bigcup_{i \leq i_1} \mathcal{V}_i$;
- (2) For each $i \geq j_k$, if $U \in \mathcal{W}_i$, then $U \notin \bigcup_{i \leq i_k} \mathcal{V}_i$;
- (3) Each element of \mathcal{W}_{j_k} is contained in $\bigcup_{i_k < i \leq i_{k+1}} \mathcal{V}_i$.

Each element of \mathcal{W}_{j_k} has d^* -diameter less than ε_{i_k} and $\varepsilon_{i_k} \leq \frac{1}{k+1}$. Thus, by definition of d^* , each element of \mathcal{W}_{j_k} is a subset of an element of \mathcal{U}_k^* , each of which in turn is a subset of an element of \mathcal{U}_k . For each k , for each element W of \mathcal{W}_{j_k} choose a $U \in \mathcal{U}_k$ with $W \subseteq U$ and let \mathcal{J}_k be the finite set of such chosen W 's.

Then, for each $y \in Y$ and for all but finitely many k we have $y \in \bigcup \mathcal{J}_k$. \square

Part IV: An example

Let Y be a subspace of X . With Ω denoting the ω -covers of X and Γ_Y denoting the γ -covers of Y by sets open in X , we say that Y is a relative γ -set in X if the selection hypothesis $\mathbf{S}_1(\Omega, \Gamma_Y)$ holds. This property was studied in [11] where it is shown that:

- (1) Every γ -set is a relative γ -set in each space of which it is a subspace;
- (2) If Y is a relative γ -set in X and $Z \subset Y$, then Z is a relative γ -set in X ;
- (3) Under the Continuum Hypothesis (CH) there are relative γ -sets which are not absolute γ -sets.

We shall now show, using CH, how to obtain a relative γ -subset Y of the Cantor set such that Y does not have the absolute Menger property $\mathbf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$. This in particular will show that even the most restrictive relative covering property is not related to the least restrictive absolute covering property. The construction is a small modification of the construction given in the proof of Theorem 1 of [6], in that we consider only ω -covers of the superspace, while [6] considers also families that need not be ω -covers of the superspace. Using the notation from [6], first observe that Lemma 1.1 of that paper is also true in the following form:

Lemma 17 [6, Lemma 1.1]. *Let X be an infinite set of positive integers and let $(\mathcal{U}_n: n \in \mathbb{N})$ be a sequence of ω -covers of $2^{\mathbb{N}}$. Let C be a countable subset of $2^{\mathbb{N}}$. Then there is a sequence $(U_n: n \in \mathbb{N})$ and an infinite subset Y of X such that:*

- (1) For each n , $U_n \in \mathcal{U}_n$; and
- (2) $C \cup Y^* \subseteq \bigcup_{m \in \mathbb{N}} \bigcap_{n > m} U_n$.

To construct our relative γ -set, proceed as follows:

Let $(f_\alpha: \alpha < \omega_1)$ enumerate ${}^{\mathbb{N}}\mathbb{N}$. Also, let $((\mathcal{U}_n^\alpha: n \in \mathbb{N}): \alpha < \omega_1)$ enumerate all sequences of countable ω -covers of $2^{\mathbb{N}}$. We shall now recursively choose infinite sets Y_α , $\alpha < \omega_1$ of positive integers such that:

- (1) For $\alpha < \beta$ we have $Y_\beta \setminus Y_\alpha$ finite; and
- (2) for each α , the enumeration function Enum_α of Y_α , eventually dominates f_α .

To begin, apply Lemma 17 to $X = \mathbb{N}$ and $(\mathcal{U}_n^0: n \in \mathbb{N})$ and choose an infinite set $Y \subset \mathbb{N}$, and a sequence $(U_n^0: n \in \mathbb{N})$ such that for each n we have $U_n^0 \in \mathcal{U}_n^0$ and $\{U_n^0: n \in \mathbb{N}\}$ is a γ -cover of Y^* . Then let Y_0 be an infinite subset of Y such that Enum_0 , the enumeration function of Y_0 , eventually dominates f_0 . Observe that $Y_0^* \subseteq Y^*$.

Let $0 < \alpha < \omega_1$ be given and assume that we have for each $\beta < \alpha$ already selected a Y_β and a sequence $(U_n^\beta: n \in \mathbb{N})$ such that:

- (1) For each n , $U_n^\beta \in \mathcal{U}_n^\beta$;
- (2) $\{U_n^\beta: n \in \mathbb{N}\}$ is a γ -cover of $\{Y_\delta: \delta < \beta\} \cup Y_\beta^*$;
- (3) For each $\delta < \beta$, $Y_\beta \in Y_\delta^*$;
- (4) For each $\delta < \alpha$, Enum_δ , the enumeration function of Y_δ , eventually dominates f_δ .

Choose first an infinite X such that for each $\beta < \alpha$ we have $X \setminus Y_\beta$ is finite. Then apply Lemma 17 to X and the countable set $C = \{Y_\beta: \beta < \alpha\}$ and the sequence $(\mathcal{U}_n^\alpha: n \in \mathbb{N})$ of ω -covers of $2^{\mathbb{N}}$ to find an infinite set $Y \subset X$, and a sequence $(U_n^\alpha: n \in \mathbb{N})$ such that for each n we have $U_n^\alpha \in \mathcal{U}_n^\alpha$, and $\{U_n^\alpha: n \in \mathbb{N}\}$ is a γ -cover of $C \cup Y^*$. Then choose an infinite set $Y_\alpha \subset Y$ such that Enum_α , the enumeration function of Y_α , eventually dominates f_α . Since $Y_\alpha^* \subseteq Y^*$, we see that $\{U_n^\alpha: n \in \mathbb{N}\}$ is a γ -cover of $\{Y_\beta: \beta < \alpha\} \cup Y_\alpha^*$.

Thus, we can recursively choose Y_α and $(U_n^\alpha: n \in \mathbb{N})$, $\alpha < \omega_1$, so that the four recursive conditions are met at each α .

Finally put $\mathcal{Y} = \{Y_\alpha: \alpha < \omega_1\}$. Then \mathcal{Y} is the required relative γ -set in $2^{\mathbb{N}}$. We must see that \mathcal{Y} does not have the Menger property $\text{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$. To see this, note that the function $F: \mathcal{Y} \rightarrow {}^{\mathbb{N}}\mathbb{N}$ defined by $F(Y_\alpha) = \text{Enum}_\alpha$ is continuous. Moreover, by construction $F[\mathcal{Y}]$ is a dominating family in ${}^{\mathbb{N}}\mathbb{N}$. By a theorem of Hurewicz, \mathcal{Y} does not have the Menger property. \square

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