AN INITIAL SEGMENT PROPERTY AND SUPERCOMPACTNESS QUANTIFICATION

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1. Introduction

We work in the base theory $\mathsf{ZF} + \mathsf{DC}$, stating other hypotheses (notably AD) as needed. We present two results which seem related to both determinacy theory and inner-model theory. In §2 we consider the question of which ordinals below ω_2 represented by functions in inner-models of $L(\mathbb{R})$. We show that for inner-models resembling those obtained as direct systems of mice that these ordinals form an initial segment of ω_2. This leads, for example, to a result about $\mathrm{HOD}^{L(\mathbb{R})}$ for which we do not see a direct AD argument. In §4 we consider the question of the closure of $\boldsymbol{\Pi}^1_3$ under quantification by various supercompactness measures. This question was first raised by John Steel and arouse from inner-model theory considerations. Although we do not use inner-model theory in the proof of the main result, there seems to be a connection, and it is possible that inner-model theoretic methods may help with question (2) of §4.

Most of the notation we employ is fairly standard. By a *tree* on X we mean a subset of $X^{<\omega}$ closed under initial segment. As usual, we identify trees on $X \times Y$ with a set of pairs $(s,t) \in X^{<\omega} \times Y^{<\omega}$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$. For T a tree on $\omega \times \delta$ and $x \in \omega^\omega$, we let $T_x = \{\vec{\alpha} \in \delta^{<\omega} \colon (x \restriction \mathrm{lh}(\vec{\alpha}), \vec{\alpha}) \in T\}$ be the section of T at x. We let $T \restriction \alpha = \{(s, \vec{\alpha}) \colon \forall i\ \alpha_i < \alpha\}$. We let $T(s)$ denote the subtree $\{t \colon t \in T \wedge t < s\}$, where $t < s$ means t extends s. When T is a tree on ordinals, we abuse notation slightly and write $T(\beta)$, implicitly identifying $\mathrm{Ord}^{<\omega}$ with Ord.

2. An Initial Segment Property for Inner Models

The "boldface" theory of $L(\mathbb{R})$ under AD gives a detailed description of the cardinal structure via an inductive analysis for a certain initial segment of the Wadge hierarchy. This includes, for example, the projective sets. This theory analyzes all the measures on the various cardinals and from this gets a good coding for all the subsets of these cardinals (see [1] and [2]). On the other hand, the "lightface" theory of HOD developed using inner-model theory shows that $\mathrm{HOD}^{L(\mathbb{R})}$ up to Θ is an extender model of the form $L[\vec{E}]$. This theory has boldface consequences, for example Steel [6] showed that every regular cardinal below Θ in $L(\mathbb{R})$ is measurable. It is of interest to see how the coding of ordinals given by the boldface theory can be analyzed from the inner-model point of view. Here we make the modest step of considering the coding of ordinals below ω_2. We first recall the boldface coding of these ordinals.

Let $\mathrm{WO} \subseteq 2^\omega$ be the standard set of codes for countable ordinals. That is, $x \in \mathrm{WO}$ iff $<_x \doteq \{(m,n) \colon x(\langle m,n\rangle = 1\}$ is a wellordering of ω. WO provides codes

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for ordinals below ω_1. Trivially, if any inner model M of ZF can an $x \in \mathrm{WO}$ coding $|x| < \omega_1$, then it has codes for all $\beta < \alpha$. Thus, the set of countable ordinals coded in M is an initial segment of ω_1.

To code ordinals below ω_2, we first recall the Kunen tree. This is a tree on $\omega \times \omega_1$ such that for any $f\colon \omega_1 \to \omega_1$, there is an $x \in \omega^\omega$ with T_x wellfounded and such that $\forall \alpha \geq \omega\ (f(\alpha) < |T_x \restriction \alpha|)$. The Kunen tree exists assuming AD (see [3], [2]).

Definition 1. For $n \geq 2$, let

$$\mathrm{WO}_n = \{\langle w, x_1, \dots, x_{n-1}\rangle \colon w \in \mathrm{WO} \wedge \forall i\ T_{x_i} \text{ is wellfounded}\}.$$

Let $\mathrm{WO}_\omega = \bigcup_n \mathrm{WO}_n$.

WO_n provides codes for ordinals below ω_n as follows. Let $z = \langle w, x_1, \dots, x_{n-1}\rangle \in \mathrm{WO}_n$. Define $f\colon (\omega_1)^{n-1} \to \omega_1$ by $f(\alpha_1, \dots, \alpha_{n-1}) = |T_{x_{n-1}} \restriction \alpha_{n-1}(\beta_{n-1})|$, where $\beta_{n-1} = |T_{x_{n-2}} \restriction \alpha_{n-2}(\beta_{n-3})|$, etc., and where $\beta_1 = |w|$. The ordinal coded by z, which we denote by $|z|$, is $[f]_{W_1^{n-1}}$, that is, the ordinal represented by f in the ultrapower by the measure W_1^{n-1}.

From the property of the Kunen tree, the normality of the measure W_1^1, and the fact that $j_{W_1^{n-1}}(\omega_1) = \omega_n$, it follows that every ordinal below ω_n is coded by some $z \in \mathrm{WO}_n$. Thus, WO_n functions as an analog of WO for the cardinals ω_n. Note that WO is $\boldsymbol{\Pi}^1_1$, while WO_n is $\boldsymbol{\Pi}^1_2$. WO_n plays a central role in the theory of the ω_n under AD just as WO does for ω_1. We refer the reader to [2] for some examples of this.

Since inner-models M of $L(\mathbb{R})$ satisfying AC can have only countably many reals, it is not reasonable to expect them to contain codes for many ordinals below ω_2, that is, $M \cap \mathrm{WO}_2$ is countable. A more natural requirement is obtained by considering the representing functions f directly. This leads to the following definition.

Definition 2. Let $M \subseteq L(\mathbb{R})$ be an inner-model of ZFC. We say M has the *initial segment property* if the set of $\alpha < \omega_2$ such that there is an $f\colon \omega_1 \to \omega_1$ in M with $\alpha = [f]_{W_1^1}$ forms an initial segment of ω_2.

Unlike the situation for ω_1, not every inner-model of $L(\mathbb{R})$ has the initial segment property. However, we show that many natural inner-models from the point of view of inner-model theory do have the initial segment property. For example, M could be $L[\mathcal{U}]$ where $\mathcal{U}$ is a measure on ω_1^V, or the limit of the directed system based on an iterable n-Woodin cardinal model, or M could be $\mathrm{HOD}^{L(\mathbb{R})}$ itself. The intention is that M can be obtained as the direct limit of a system based on a countable iterable model of a definite strength, i.e., satisfying a minimality condition. The directed system itself plays no role in the following arguments, so we abstract the properties of M we need in the following definition.

Definition 3. A inner-model $M \subseteq L(\mathbb{R})$ is *reasonable* if it satisfies the following.

(1) $M = L[\vec{E}]$ for some coherent extender sequence $\vec{E}$.
(2) ω_1^V is the critical point of the least extender E_0 on $\vec{E}$.
(3) $E_0 \cap M = W_1^1 \cap M$.
(4) M and $j_{W_1^1}(M)$ have a successful comparison in $L(\mathbb{R})$.
(5) There is a formula φ such that $M \models \varphi$ but $N \models \neg\varphi$ where $N = L[\vec{E} \restriction \alpha]$ and $\alpha < \mathrm{lh}(\vec{E})$.

Actually both the iterability and minimality requirement can be weakened a great deal, but the weakening is technical and difficult to axiomatize in this way, this issue is dealt with in the next section.

We now state our initial segment result.

Theorem 4 (AD)**.** Every reasonable inner model $M \subseteq L(\mathbb{R})$ has the initial segment property.

Proof. Let $M \subseteq L(\mathbb{R})$ be reasonable. Let $M' = j(M)$, where j is the embedding from the least measurable E_0 on the extender sequence. Thus, M' is the internal ultrapower (i.e., formed in M) obtained by using the least extender (measure) on the sequence. Let $N = j_{W_1^1}(M)$ be the external ultrapower, formed in $L(\mathbb{R})$, of M by the measure W_1^1 on ω_1^V. This ultrapower is formed using functions $f\colon \omega_1 \to M$ in $L(\mathbb{R})$. From property (3) of reasonableness it follows that there is an embedding $k\colon M' \to N$ given by $k([f]_{E_0}) = [f]_{W_1^1}$. Let $\kappa = \mathrm{crit}(k)$.

Clearly $\kappa > \omega_1$. Note that there is in M, and hence in $L(\mathbb{R})$ a wellordered sequence of length $j(\omega_1^M)$ of distinct subsets of ω_1^V. Since $\omega_2^{L(\mathbb{R})}$ is measurable, it follows that this sequence has length $< \omega_2^{L(\mathbb{R})}$. Since $k(j(\omega_1^M)) = \omega_2^{L(\mathbb{R})}$, it follows that $\kappa < \omega_2^{L(\mathbb{R})}$. Note that κ can be characterized as the least ordinal below $\omega_2^{L(\mathbb{R})}$ which is not of the form $[f]_{W_1^1}$ for some $f \in M$. The initial segment property of M is therefore equivalent to the statement that $\kappa = j(\omega_1^V)$.

Suppose $\alpha < j(\omega_1^V)$. Let D be a proper class of $L(\mathbb{R})$-cardinals $> \Theta^{L(\mathbb{R})}$ fixed by the embeddings j, $j_{W_1^1}$, and k. D will automatically be fixed by the comparisons that come up in the following argument.

Claim 5. $\alpha = \tau^{M'}(\beta_1, \ldots, \beta_m, \omega_1, \delta_1, \ldots, \delta_n)$ for some term τ and ordinals $\beta_1, \ldots, \beta_m < \omega_1^V$, and $\delta_1, \ldots, \delta_n \in D$.

Proof. Since $\alpha < j(\omega_1^V)$, it is of the form $j(f)(\omega_1^V)$ for some $f \in M$. Since the ordinals in D are fixed by j, it suffices to show that f is definable in M from finitely many ordinals below ω_1^V and finitely many ordinals in D. More generally, we show every $A \subseteq \omega_1^V$ in M is so definable in M. Let $\bar{M}$ be the transitive collapse of $\mathrm{Hull}^M(\omega_1 \cup D)$. Since $\bar{M}$ is still iterable (being the collapse of a substructure of the iterable M) we can compare M and $\bar{M}$. Let $\pi_0\colon M \to P$, $\pi_1\colon \bar{M} \to Q$ be the comparison maps. Since both P and Q are minimal satisfying φ, it follows that $P = Q$. Since $\mathrm{crit}(\pi_0)$, $\mathrm{crit}(\pi_1) \geq \omega_1^V$, it follows that $\mathcal{P}(\omega_1^V) \cap M' = \mathcal{P}(\omega_1^V) \cap P = \mathcal{P}(\omega_1^V) \cap Q = \mathcal{P}(\omega_1^V) \cap \bar{M}$. Thus, $A \in \bar{M}$. Let $B \in \mathrm{Hull}^M(\omega_1 \cup D)$ be such that $\pi(B) = A$, where $\pi\colon \mathrm{Hull}^M(\omega_1^V \cup D) \to \bar{M}$ is the collapse map. Since the hull contains ω_1^V, $B = A$. But, B is definable in M from ordinals below ω_1^V and ordinals in D, from the definition of $\mathrm{Hull}^M(\omega_1^V \cup D)$. □

So, if $\alpha < j(\omega_1^V)$, then $\alpha = \tau^{M'}(\beta_1, \ldots, \beta_m, \omega_1^V, \delta_1, \ldots, \delta_n)$. Thus, $k(\alpha) = \tau^N(\beta_1, \ldots, \beta_m, \omega_1^V, \delta_1, \ldots, \delta_n)$ since $\mathrm{crit}(k) > \omega_1^V$ and the δ_i are fixed by k. We now compare the models M' and N. This is possible by property (4) since if $\mathcal{T}$ and $\mathcal{U}$ are the trees arising from the comparison of M and N, then the first extender used on the M side is E_0 and thus $M' = M_1^{\mathcal{T}}$ and the comparison will never return to $M_0^{\mathcal{T}}$ and thus $\mathcal{T}$ can essentially be viewed as a comparison of M' with N. Let $\sigma_0\colon N \to \mathcal{R}$, $\sigma_1\colon M' \to \mathcal{S}$ be the comparison maps. Since both M' and N satisfy the minimality condition φ, it follows that $\mathcal{R} = \mathcal{S}$. Note that $\mathrm{crit}(\sigma_0)$, $\mathrm{crit}(\sigma_1) > \omega_1^V$. Since the δ_i are fixed by both σ_0 and σ_1, we have that

$\sigma_1(\alpha) = \sigma_1(\tau^{M'}(\vec{\beta}, \omega_1^V, \vec{\delta}) = \tau^{\mathcal{S}}(\vec{\beta}, \omega_1^V, \vec{\delta}) = \tau^{\mathcal{R}}(\vec{\beta}, \omega_1^V, \vec{\delta}) = \sigma_0(\tau^N(\vec{\beta}, \omega_1^V, \vec{\delta}) = \sigma_0(k(\alpha))$. However, $\alpha < \mathrm{crit}(\sigma_1)$ and $k(\alpha) < \mathrm{crit}(\sigma_0)$ since $\mathrm{crit}(\sigma_1) \geq \omega_1^V$ and $\mathrm{crit}(\sigma_0) \geq \omega_2^V = k(\omega_1^V)$. Thus, $\alpha = k(\alpha)$. This show that $\mathrm{crit}(k) = j(\omega_1^V)$ and completes the proof of the theorem. □

Since $\mathrm{HOD}^{L(\mathbb{R})} \restriction \boldsymbol{\delta}_1^2$ is a reasonable inner-model (a result of Steel), we have the following corollary.

Corollary 6 ($\mathsf{AD} + V = L(\mathbb{R})$)**.** If $f : \omega_1 \to \omega_1$ is in HOD, and $\beta < [f]_{W_1^1}$, then there is a $g \in \mathrm{HOD}$ with $\beta = [g]_{W_1^1}$.

That is, the ordinals below ω_2 which are representable by functions in HOD form an initial segment of ω_2. Although this corollary is ostensibly a result about $L(\mathbb{R})$ under AD, we don't see a proof not using inner-models.

3. Reasonable inner models associated with a pointclass

Reasonable inner models arise as limits of directed systems of mice which occur naturally in Woodin's calculation of HOD in models of determinacy (see for example [6]) as well as in the core model induction. The existence of these structures is outlined in this section. This is not intended to be a complete discussion of this issue, but rather an outline to indicate how the supercompactness measures enter in to give property 4 of reasonableness.

Let Γ be an *inductive-like* pointclass, i.e. Γ is scaled, ω-parametrized, and closed under real quantification. Let T_Γ be a tree for a scale on a Γ-universal set U_Γ. Notice T_Γ is a tree on $\omega \times \delta_\Gamma$ where $\delta_\Gamma = \sup\{\alpha : \alpha$ is the length of a Δ prewellorder of $\mathbb{R}\}$. For the remainder of this section fix Γ, U_Γ, and T_Γ as above.

By a theorem of Harrington and Kechris [**?**] whenever a is countable and transitive, $C_\Gamma(a) = \mathcal{P}(a) \cap L(T_\Gamma, a)$ where for a real x, $C_\Gamma(x)$ is the largest countable Γ set and $b \in C_\Gamma(a)$ if and only if $\forall^* g \in a^\omega\, (\, b_g \in C_\Gamma(a_g)\,)$ where b_g and a_g are the obvious codings of b and a from g and $\forall^*$ means for "*comeagerly many*". A countable transitive set a is called Γ*-full* if a is a rank initial segment of $L(T_\Gamma, a)$, that is, a already contains all bounded subsets of a that are constructed from a relative to T_Γ. We will assume that Γ is captured by mice (MSC_Γ) which means that $L(T_\Gamma, a) \cap \mathcal{P}(a) \subseteq \mathrm{Lp}^\Gamma(a)$ for any countable transitive a where $\mathrm{Lp}^\Gamma(a)$ is the union of all a-premice M with $\rho_\omega(M) = a$ which have countable iteration strategies in Γ. Thus a is Γ-full just in case no a-mouse with iteration strategy in Γ adds a bounded subset to a.

Call a set A of reals countably captured over Γ if $A \cap \sigma \in L(T_\Gamma, \sigma)$ for all $\sigma \subseteq \mathbb{R}$ countable. Let Γ' be the be the boldface pointclass of sets countably captured over Γ. Woodin has shown that $\Gamma = \Sigma_1^2(\Gamma')$ and if δ_Γ is not the largest Suslin cardinal, then every set in $\mathrm{OD}_{\Gamma'}$ has a scale all of whose norms are in $\mathrm{OD}_{\Gamma'}$. Thus there is a sequence $\mathcal{A} = \langle A_i : i \in \omega \rangle$ such that $\mathcal{A}$ is closed under complements, $A_0 = U_\Gamma$ is a universal Γ set, and each A_i has a scale all of whose norms are in $\mathcal{A}$. Such a set is called a *self-justifying system* for Γ. Fix such an $\mathcal{A}$ for the remainder of this section. Notice that $\delta_{\Gamma'}$ is the first Suslin cardinal past δ_Γ and has cofinality ω.

For $A \subseteq \mathbb{R}$ and countable γ define the *standard* $\mathrm{Col}(\omega, \gamma)$*-term for* A by

$$\tau_{A,\gamma} = \{(p, \sigma) : \sigma \subseteq \gamma^{<\omega} \times \omega^{<\omega} \text{ is a term for a real } \wedge \forall^* g \subseteq \gamma^\omega\, (p \subseteq g \to \sigma_g \in A)\}$$

It follows from the definition of Γ' that if $A \in \Gamma'$ and M is countable and transitive and Γ-full, then for $\gamma \in M$, $M \cap \tau_{A,\gamma} \in M$.

Call a premouse M a Γ-Woodin if

(1) $M \models$ "δ_M is Woodin + Ord $= \delta_M^{+\omega}$".
(2) M is Γ-full.
(3) M is Γ-small in that $L[T_\Gamma, M|\alpha] \models$ "$\forall \alpha < \delta_M$ (α is not Woodin)". (This is the more general notion of *minimality* that replaces property 5 of reasonableness.)

For M a Γ-Woodin set $\tau^M_{i,j} = \tau_{A_i, \delta_M^{+j}} \cap M$. One important consequence of the above definitions is that if M is a Γ-Woodin and $\{\tau^M_{i,j} : i, j < \omega\} \subseteq X \prec M$, $\pi : \bar{M} \to M$ is the inverse of the collapse embedding, then $\bar{M}$ is a Γ-Woodin and $\pi^{-1}(\tau^M_{i,j}) = \tau^{\bar{M}}_{i,j}$, that is, the term relations collapse correctly. In the other direction if M is full, $\pi : M \to N$, and $\pi(\tau^M_{i,j}) = \pi^N_{i,j}$, then N is full. (It is not generally true that N being closed under term relations for $\mathcal{A}$ implies that N is full, but this is true if N is an iterate of a full M.)

Let M be a Γ-Woodin and let $\mathcal{T}$ be an iteration tree on M. Call a limit length normal iteration tree $\mathcal{T}$ Γ-*maximal* if $L(T_\Gamma, M(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is Woodin". If $\mathcal{T}$ is not maximal, then there is at most one cofinal wellfounded branch b such that $Q_\Gamma(M(\mathcal{T})) \trianglelefteq M^{\mathcal{T}}_b$ where $Q_\Gamma(M(\mathcal{T}))$ is the first $M(\mathcal{T})$-mouse N such that $\rho_\omega(N) = \delta(\mathcal{T})$, N has iteration strategy in Γ, and N definably kills the Woodiness of $\delta(\mathcal{T})$. Call an iteration tree Γ-guided if at each limit $\lambda \leq \mathrm{lh}(\mathcal{T})$, $[0, \lambda]_{\mathcal{T}}$ is the unique cofinal wellfounded branch b such that $Q_\Gamma(M(\mathcal{T})) \trianglelefteq M^{\mathcal{T}}_b$. If $\mathcal{T}$ is Γ-guided and maximal then there is at most one cofinal wellfounded branch that correctly moves all of the term relations from $\mathcal{A}$. These two facts lead to the notion of M being iterable by a Γ-guided strategy. (The choice of $\mathcal{A}$ is not important so this really is a property of Γ.) Under our assumptions there is a Γ-Woodin with a unique Γ-guided iteration strategy Σ_M. The Γ-guided strategies have the Dodd-Jensen property so if $i, i' : M \to N$ arise from two different Γ-guided iterations, then $i = i'$. For N an iterate of M simply write $i_{M,N}$ for the unique iteration map from M to N.

The directed system we are after is $\mathcal{D}^\Gamma$ consisting of all Γ-Woodins M with Γ-guided countable iteration strategies and for $M, N \in \mathcal{D}^\Gamma$ set $M \prec_\Gamma N$ if and only if N is an iterate of M. This system is countably closed so its limit M^Γ_∞ is wellfounded. It is also turns out that $\delta^\Gamma_\infty = \lambda$ where $\lambda = \delta_{\Gamma'}$ is the next Suslin cardinal after δ_Γ. The *reasonable* inner model we associate to the pointclass Γ is M^Γ_∞. Assume $\Theta_0 < \Theta$ and $\Gamma = \Sigma^2_1$, then $\Gamma' = \mathcal{P}_{\Theta_0}(\mathbb{R})$ and HOD $\restriction \Theta_0 = M^{\Sigma^2_1}_\infty \restriction \Theta_0$. This is true if $\Theta_0 = \Theta$ as well, but some modifications need to be made in this case.

Let κ be an arbitrary Suslin cardinal. Let $g \subseteq \mathrm{Col}(\omega, \kappa)$ be $L(\mathbb{R})$-generic and let $j_g : L(\mathbb{R}) \to L(\mathbb{R}_g)$ be the corresponding generic embedding. Let j_κ be the associated embedding from the $\mathcal{P}_{\omega_1}(\kappa)$ supercompactness measure. Then Γ moves naturally to $j_g(\Gamma)$ via $j_\kappa(T_\Gamma)$ and $j_\kappa[\mathcal{D}^\Gamma] \subseteq \mathcal{D}^{j_\kappa(\Gamma)}$, moreover if $\kappa \geq \lambda$, then $M^\Gamma_\infty \in \mathcal{D}^{j_\kappa(\Gamma)}$. More generally this holds for intermediate extensions as well so that if $\kappa \geq \lambda > \omega_1$, then $M^{j_{\omega_1}(\Gamma)}_\infty = j_{W^1_1}(M^\Gamma_\infty) \in \mathcal{D}^{j_\kappa(\Gamma)}$. This gives property 4 of reasonableness for M^Γ_∞.

The following two facts and the proof of the first are copied almost word for word from [7]. The proof is included just to indicate how things carry over almost word for word. The proof of the second fact requires a bit more work and is left to the reader.

Lemma 7. Let $\kappa < \delta_\infty$ be regular in M_∞, then exactly one of the following hold:

(1) $M_\infty \models$ "κ is measurable",
(2) $\mathrm{cf}^{L(\mathbb{R})}(\kappa) = \omega$

Proof. Fix $\kappa < \delta_\infty$ regular in M_∞ and let $\bar{M}$ be in the system with $\pi_{\bar{M},M_\infty}(\bar{\kappa}) = \kappa$. If $\bar{\kappa}$ is not measurable in $\bar{M}$, then $\bar{\kappa}$ is a continuity point of $\pi_{\bar{M},M_\infty}$ and thus κ has cofinality ω. This is because $\pi_{\bar{M},M_\infty}$ is essentially an iteration map and all regular non-measurable cardinals of $\bar{M}$ are continuity points of all simple iteration maps on M.

Suppose that $\bar{\kappa}$ is measurable in M, then of course $M_\infty \models$ "κ is measurable" so we must show $\mathrm{cf}^{L(\mathbb{R})}(\kappa) > \omega$. Let $X \subseteq \kappa$ be countable in $L(\mathbb{R})$ and cofinal in κ. Since the directed system is countably complete, there is $\hat{M}$ above $\bar{M}$ in the directed system such that $X \subseteq \pi_{\hat{M},M_\infty}[\hat{X}]$. $\hat{\kappa}$ is measurable in $\hat{M}$ so let $M' = \mathrm{ult}(\hat{M}, E)$, where E is a normal measure on $\hat{\kappa}$ in $\hat{M}$. $\pi_{\hat{M},M_\infty} = \pi_{M',M_\infty} \circ \pi_{\hat{M},M'}$ so $\pi_{M',M_\infty}[\hat{X}] \supseteq X$ so $X \subseteq \pi_{M',M_\infty}(\hat{\kappa}) < \kappa$. □

Theorem 8. Let $\kappa < \delta_\infty$ and $\mathrm{cf}^{L(\mathbb{R})}(\kappa) > \omega$ so $M_\infty \models$ "κ is measurable". Let μ is the unique order zero measure on κ in M_∞, then μ is generated by the ω-club filter (in $L(\mathbb{R})$).

These two results show that if μ is the order zero measure on $\omega_1^{L(\mathbb{R})}$ in M_∞^Γ, then $\mu = W_1^1 \cap M_\infty^\Gamma$ which is all we wanted.

4. Quantification by Supercompactness Measures

The Kechris-Martin theorem asserts that Π^1_3 is closed under ordinal quantification of length ω_ω. More precisely,

Theorem 9 (Kechris-Martin)**.** Assume AD. suppose $P \subseteq \omega^\omega \times \mathrm{WO}_\omega$ be Π^1_3 and invariant in the codes, that is, if $P(x, w)$ and $w' \in \mathrm{WO}_\omega$ with $|w'| = |w|$, then $P(x, w')$. then $A(x) \leftrightarrow \exists w \in \mathrm{WO}_\omega\ P(x, w)$ is also Π^1_3.

An immediate boldface consequence of this is the following (the intersection case follows from the coding lemma and does not need the Kechris-Martin result).

Corollary 10 (Kechris-Martin)**.** Assume AD. $\boldsymbol{\Pi}^1_3$ is closed under $< \boldsymbol{\delta}^1_3$ length unions and intersections.

If $P \subseteq \omega^\omega \times \mathrm{WO}_\omega$ is invariant in the codes, we will write $P(x, \alpha)$ to mean $P(x, w)$ for some (any) $w \in \mathrm{WO}_\omega$ with $|w| = \alpha$.

In fact, $\boldsymbol{\Pi}^1_3$ is also closed under quantification by measures on ω_ω. More precisely, we have:

Theorem 11. Let $P \subseteq \omega^\omega \times \omega_\omega$ be $\boldsymbol{\Pi}^1_3$ and invariant in the codes. Let μ be a measure on ω_ω. Then $A(x) \leftrightarrow \forall^*_\mu \alpha < \omega_\omega\ (P(x, \alpha))$ is also $\boldsymbol{\Pi}^1_3$.

Theorem 11 follows from the analysis of measures on ω_ω, which can be found, for example, in [2].

In view of theorem 11 the following are reasonable questions. Question (1) was raised by John Steel.

Question 1. Is $\boldsymbol{\Pi}^1_3$ closed under quantification by the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_\omega)$?

Question 2. Is $\mathbf{\Pi}^1_3$ closed under quantification by the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_n)$ for $n < \omega$?

To make these questions precise, let $C = \{y \colon \forall n\ (y)_n \in \mathrm{WO} - \omega\}$. We say $P \subseteq \omega^\omega \times C$ is invariant in the codes for countable sets if whenever $P(x, y)$ and $y' \in C$ and $\{|(y)_n \colon n \in \omega\} = \{|(y')_n \colon n \in \omega\}$, then $P(x, y)$. In this case we write $P(x, S)$, for $S \in \mathcal{P}_{\omega_1}(\omega_\omega)$, to mean $P(x, y)$ for some (any) $y \in C$ such that $S = \{|(y)_n| \colon n \in \omega\}$. We say that $\mathbf{\Pi}^1_3$ is closed under quantification by the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_\omega)$ if whenever $P \subseteq \omega^\omega \times C$ is invariant in the codes for countable sets, then $A(x) \leftrightarrow \forall^*_{\nu_\omega} S\ P(x, S)$ is in $\mathbf{\Pi}^1_3$, where ν_ω denotes the supercomapctness on $\mathcal{P}_{\omega_1}(\omega_\omega)$. We likewise formalize question (2), using the supercompactness measure ν_n on $\mathcal{P}_{\omega_1}(\omega_n)$.

Although the analogs of questions (1) and (2) for measures are immediately equivalent (even without knowing that the answers to both are yes), the situation for supercompactness measures is not clear. In other words, a positive answer for (2) does not immediately imply one for (1).

We show next that question (1) has a negative answer.

Theorem 12 (AD)**.** $\mathbf{\Pi}^1_3$ is not closed under quantification by the supercompactness measure on $\mathcal{P}_{\omega_1}(\omega_\omega)$.

Proof. It suffices to show that every $\mathbf{\Sigma}^1_3$ set $A \subseteq \omega^\omega$ can be written in the form $A(x) \leftrightarrow \forall^*_{\nu_\omega} S\ P(x, S)$, where $P \subseteq \omega^\omega \times C$ is $\mathbf{\Pi}^1_3$ and invariant in the codes for countable sets.

Fix a $\mathbf{\Delta}^1_3$ regular scale $\{\psi_n\}_{n \in \omega}$ on a $\mathbf{\Pi}^1_2$-complete sets P. The main lemma of the Kechris-Woodin theory of generic codes says that there is a Lipschitz continuous (i.e., a strategy for II) generic coding function $G \colon (\omega_\omega)^\omega \to \omega^\omega$. This means that for all $s = (\alpha_0, \alpha_1, \dots) \in (\omega_\omega)^\omega$ we have that $\forall n\ [(G(s))_n \in \mathrm{WO}_\omega \wedge |(G(s))_n| \leq \alpha_n]$. Also, if s enumerates an honest set $S \in \mathcal{P}_{\omega_1}(\omega_\omega)$ then $\forall n\ |(G(s))_n| = \alpha_n$. Recall that S is honest if for every $\alpha \in S$ there is an $x \in P$ with $\psi_n(x) = \alpha$ for some n, and such that for all i, $\psi_i(x) \in S$.

Since $\mathbf{\Sigma}^1_3$ equal ω_ω-Suslin under AD, there is a tree T on $\omega \times \omega_\omega$ such that $A = p[T]$. Fix also a universal $\mathbf{\Sigma}^1_1$ set $U \subseteq \omega^\omega \times \omega^\omega$.

Note that for $S \in \mathcal{P}_{\omega_1}(\omega_\omega)$, $A_S \doteq p[T \restriction S] \in \mathbf{\Sigma}^1_1$, since S is countable and ω-Suslin equals $\mathbf{\Sigma}^1_1$.

Consider the following ordinal game $\mathcal{G}$:

I	α_0		α_2		α_4		$\dots$
II		α_1		α_3		α_5	$\dots$
		$x(0)$		$x(1)$		$x(2)$	$\dots$

where I and II play ordinals $\alpha_i < \omega_\omega$ and II also plays integers $x(i)$, building a real $x \in \omega^\omega$. Let $s = (\alpha_0, \alpha_1, \dots)$ be the sequence of ordinals played, and $S = \{\alpha_0, \alpha_1, \dots\}$ the set enumerated by s. II wins the run of the game iff $U_x = A_{S'}$, where $U_x = \{y \colon U(x, y)\}$ is the $\mathbf{\Sigma}^1_1$ set coded by x. Hers, $S' = \{|(y)_n| \colon n \in \omega\}$, where $y = G(s)$.

The game $\mathcal{G}$ is determined since it is a Suslin, co-Suslin ordinal game (cf. []). This is because the payoff for $\mathcal{G}$ depends only on the reals x and y described above, and the payoff condition is a Suslin and co-Suslin set of (x, y) (in fact, it is projective).

Clearly I cannot have a winning strategy σ for $\mathcal{G}$. For II could defeat σ by enumerating an honest set S closed under σ (for all integer moves by II) and playing also $x \in \omega^\omega$ such that $U_x = A_S$.

Fix a winning strategy τ for II in $\mathcal{G}$. If $s = (\alpha_0, \alpha_1, \dots)$, then we let $\tau(s)$ denote the real x played by II following τ when I plays $\alpha_0, \alpha_1, \dots$. Note that if s enumerates an honest set S closed under τ, then $U_{\tau(s)} = A_S$.

Define $R \subseteq \omega^\omega \times \omega^\omega$ by:

$$R(x, y) \leftrightarrow \forall n\ (y)_n \in \mathrm{WO}_\omega \wedge \forall z\ [(\forall i (z)_i \in \mathrm{WO}_\omega \wedge \{|(z)_i|\} = \{|(y)_i|\}) \to \\ \forall w\ \{(w = \tau(|(z)_0|, |(z)_1|, \dots) \to x \in U_w\}]$$

From the coding lemma we have that τ is $\mathbf{\Sigma}^1_3$ in the codes. That is, for each n the relation

$$S(z_0, z_1, x(0), \dots, z_{2n+1}, x(n)) \leftrightarrow (z_0, \dots, z_{2n+1} \in \mathrm{WO}_\omega \wedge \\ (|z_0|, |z_1|, x(0), \dots, |z_{2n+1}|, x(n)) \text{ is according to } \tau)$$

is $\mathbf{\Sigma}^1_3$. Thus the relation $\forall n\ (z)_n \in \mathrm{WO}_\omega \wedge w = \tau(|(z)_0|, |(z)_1|, \dots)$ is also $\mathbf{\Sigma}^1_3$. From this, a straightforward computation shows that $R \in \mathbf{\Pi}^1_3$.

We have

$$\begin{aligned} A(x) &\leftrightarrow x \in p[T] \\ &\leftrightarrow \forall^*_{\nu_\omega} S\ x \in p[T \restriction S] \\ &\leftrightarrow \forall^*_{\nu_\omega} S\ [x \in U_{\tau(\vec{\alpha})} \text{ for any } \vec{\alpha} \text{ enumerating } S] \\ &\leftrightarrow \forall^*_{\nu_\omega} S\ R(x, S) \end{aligned}$$

The third equivalence follows from the fact that almost all S are honest and closed under τ. The last equivalence follows from the definition of R. □

As we mentioned before, theorem 12 does not rule out the possibility that $\mathbf{\Pi}^1_3$ is closed under quantification by the supercompactness measures ν_n on $\mathcal{P}_{\omega_1}(\omega_n)$ for some (or every) n. This closure would, in fact, imply the Kechris-Martin theorem. To see this, let $P \subseteq \omega^\omega \times \mathrm{WO}_\omega$ be $\mathbf{\Pi}^1_3$ and invariant in the codes. Define $P_n \subseteq \omega^\omega \times C$ by $P_n(x, y) \leftrightarrow \forall i\ (y)_i \in \mathrm{WO}_n \wedge \exists i\ P(x, (y)_i)$. then $P_n \in \mathbf{\Pi}^1_3$ and is invariant in the codes for countable sets. Let $A(x) \leftrightarrow \exists \alpha\ P(x, \alpha)$. Clearly,

$$\begin{aligned} A(x) &\leftrightarrow \exists n\ \forall^*_{\nu_n} S \in \mathcal{P}_{\omega_1}(\omega_n)\ \exists \alpha \in S\ P(x, \alpha) \\ &\leftrightarrow \exists n\ \forall^*_{\nu_n} S \in \mathcal{P}_{\omega_1}(\omega_n)\ P_n(x, S), \end{aligned}$$

and so we have $A \in \mathbf{\Sigma}^1_3$. So, a positive answer to question (2) would be a strengthening of the Kechris-Martin theorem.

Although we do not know the answer to question (2), there is some evidence that there might be a positive answer. First, we have the following.

Proposition 13. $\mathbf{\Pi}^1_3$ is closed under supercompactness quantification over $\mathcal{P}_{\omega_1}(\omega_2)$.

Proof. Let S^1_1 denote the ω-cofinal normal measure on ω_2. We define a measure μ on $\mathcal{P}_{\omega_1}(\omega_2)$ as follows.

$$\mu(A) = 1 \leftrightarrow \forall^*_{S^1_1}\alpha \text{ for some (any) bijection } \pi\colon \alpha \to \omega_1\ \forall^*_{W^1_1}\beta\ (\pi[\beta] \in A).$$

Note that for any $\alpha < \omega_2$, if π_1, π_2 are both bijections from α to ω_1, then for almost all $\beta < \omega_1$ respect to W^1_1 we have $\pi_1[\beta] = \pi_2[\beta]$. [This holds for any β closed under $\pi_2 \circ \pi_1^{-1}$ and $\pi_1 \circ \pi_2^{-1}$.] Thus, the quantifier after the $\forall^*_{S^1_1}\alpha$ in the above definition could be either $\exists\pi$ or $\forall\pi$.

Clearly μ is a fine measure on $\mathcal{P}_{\omega_1}(\omega_2)$. We next claim that μ is also normal. Suppose $F\colon \mathcal{P}_{\omega_1}(\omega_2) \to \omega_2$ is given and $F(S) \in S$ for all S. Thus, $\forall^*_{S^1_1}\alpha\ \exists\pi\ \forall^*_{W^1_1}\beta\ \exists\gamma < \beta\ (\pi(\gamma) = F(\pi[\beta]))$. Since W^1_1 is normal, we have $\forall^*_{S^1_1}\alpha\ \exists\pi\ \exists\gamma < \omega_1\ \forall^*_{W^1_1}\beta\ (\pi(\gamma) = F(\pi[\beta]))$. That is, $\forall^*_{S^1_1}\alpha\ \exists\delta < \alpha\ \exists\pi\ \forall^*_{W^1_1}\beta\ (\delta = \pi[\beta])$. Since S^1_1 is also normal we have $\exists\delta < \omega_2\ \forall^*_{S^1_1}\alpha\ \exists\pi\ \forall^*_{W^1_1}\beta\ (\delta = \pi[\beta])$. That is, $\exists\delta < \omega_2\ \forall^*_{\mu}S\ (\delta = F(S))$. Thus, μ is normal.

By Woodin's theorem [8] on the uniqueness of supercompactness measures, ν_2 is the unique fine normal measure on $\mathcal{P}_{\omega_1}(\omega_2)$. Thus, $\mu = \nu_2$.

Suppose now $P \subseteq \omega^\omega \times C$ is invariant in the codes for countable sets and $P(x, y)$ implies $\forall n\ (y)_n \in \mathrm{WO}_2$. Let $A(x) \leftrightarrow \forall^*_{\nu_2}S\ P(x, S)$.

Recall that from AD that there is a $\boldsymbol{\Delta}^1_3$ coding of the subsets of ω_ω [5] (a proof using WO_ω is given in [2]). That is, there is an onto map $\rho\colon \omega^\omega \to \mathcal{P}(\omega_\omega)$ such that for any $\alpha < \omega_\omega$, $\{x\colon \alpha \in \rho(x)\} \in \boldsymbol{\Delta}^1_3$.

We have:

$$A(x) \leftrightarrow \forall^*_{S^1_1}\alpha\ \forall\pi\ \forall^*_{W^1_1}\beta\ P(x, \pi[\beta]).$$

Let $Q(x, z) \leftrightarrow z \in \mathrm{WO}_2 \wedge \forall$ bijections $\pi\colon |z| \to \omega_1\ \forall^*_{W^1_1}\beta\ P(x, \pi[\beta])$. Thus, Q is invariant in the codes for z and

$$A(x) \leftrightarrow \forall^*_{S^1_1}\alpha\ Q(x, \alpha, \pi[\beta]).$$

So, it suffices by theorem 11 to show that $Q \in \boldsymbol{\Pi}^1_3$. Let $B(w, z) \leftrightarrow z \in \mathrm{WO}_2 \wedge \rho(w)$ is a bijection from $|z|$ to ω_1. Since ρ is a $\boldsymbol{\Delta}^1_3$ coding and since $\boldsymbol{\Delta}^1_3$ is closed under $< \boldsymbol{\delta}^1_3$ length unions and intersections (a result of Martin; see [2]), it follows that $B \in \boldsymbol{\Delta}^1_3$. We have:

$$\begin{aligned} Q(x, z) &\leftrightarrow z \in \mathrm{WO}_2 \wedge \forall w\ [(B(w, z) \to \forall^*_{W^1_1}\beta\ P(x, (\rho(w)[\beta])] \\ & z \in \mathrm{WO}_2 \wedge \forall w\ [(B(w, z) \to \forall^*_{W^1_1}\beta\ S(x, w, z, \beta) \end{aligned}$$

where

$$\begin{aligned} S(x, w, z, u) \leftrightarrow{} & (z \in \mathrm{WO}_2) \wedge B(w, z) \wedge u \in \mathrm{WO} \\ & \wedge \forall v\ [(v \in C \wedge \{|(v)_i|\} = \rho(w)[|u|]) \to P(x, v)]. \end{aligned}$$

S is invariant in the codes for u, and $S \in \boldsymbol{\Pi}^1_3$ using again the facts that ρ is a $\boldsymbol{\Delta}^1_3$ coding and the closure of $\boldsymbol{\Delta}^1_3$ under $< \boldsymbol{\delta}^1_3$ length unions and intersections. From theorem 11 it follows that $Q \in \boldsymbol{\Pi}^1_3$. □

The proof of proposition 13 shows that the supercompactness measure ν_2 is close to being an ordinal measure. The first author has shown that the supercompactness measures ν_n also behave like ordinal measures in a certain sense according to the next proposition. We let o.t.(A) denote the order-type of A for $A \subseteq \mathrm{Ord}$.

Proposition 14 (AD)**.** Suppose $F\colon \mathcal{P}_{\omega_1}(\omega_n) \to \omega_1$. Then there is a function $f\colon (\omega_1)^n \to \omega_1$ such that

$$\forall^*_{\nu_n} S \in \mathcal{P}_{\omega_1}(\omega_n)\ [F(S) = f(\mathrm{o.t.}(S \cap \omega_1), \mathrm{o.t.}(S \cap \omega_2), \ldots, \mathrm{o.t.}(S \cap \omega_n))].$$

Question (2) remains open, however, for $n > 2$.

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