1 Link to Back Issues of the Diary

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2 3/14/07 - Wednesday - Day 33

1 Beginnings of Laplace Transforms.

2 If $f$ is a function, then $L(f)$, given by

$$L(f) = \int_0^\infty f(t)e^{-st} \, dt,$$

is the celebrated Laplace Transform of $f$.

3 $L(1) = \frac{1}{s}$ for $s > 0$.

4 $L(e^{At}) = \frac{1}{s - A}$ for $s > A$.

3 3/16/07 - Friday - Day 34 - Test #2

1 Test #2: click here for a large-file answer key.
4  3/19/07 - Monday - Day 35

1  L'Hôpital's-Rule and Integration-by-Parts Review: for positive-integer $n$,

$$\mathcal{L} (t^n) = \frac{n}{s} \mathcal{L} (t^{n-1})$$

and so

$$\mathcal{L} (t^n) = \frac{n!}{s^{n+1}}$$

2  Boring but necessary: the linearity:

$$\mathcal{L} (\alpha f + \beta g) = \alpha \mathcal{L} (f) + \beta \mathcal{L} (g)$$

This lets us get stuff like:

$$\mathcal{L} (3t^4 + 5e^{-st}) = 3 \mathcal{L} (t^4) + 5 \mathcal{L} (e^{-st}) = 3 \left( \frac{4!}{s^5} \right) + 5 \left( \frac{1}{s + 8} \right)$$

3  By parts again:

$$\mathcal{L} (f') (s) = s \mathcal{L} (f) (s) - f(0),$$

provided $\mathcal{L} (f')$ exists and $\lim_{t \to \infty} e^{-st} f(t) = 0$.

4  Lookahead: we’ll solve the initial-value problem

$$y' + 4y = e^{st} \text{ with } y(0) = 7$$

using Laplace transforms and partial fractions.
5 Laplace Transforms Particular Functions

This table is stuff you need to have on board your gray matter for the rest of the semester. Rote memorization would not be a bad reaction to this, though the derivations of the various formulas are fairly straightforward.

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st} , dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}$ if $s &gt; 0$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$ if $s &gt; 0$ and $n$ is a positive integer</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$ if $s &gt; a$</td>
</tr>
<tr>
<td>$\cos(\beta t)$</td>
<td>$\frac{s}{s^2 + \beta^2}$</td>
</tr>
<tr>
<td>$\sin(\beta t)$</td>
<td>$\frac{\beta}{s^2 + \beta^2}$</td>
</tr>
<tr>
<td>$\cosh(\beta t)$</td>
<td>$\frac{s}{s^2 - \beta^2}$</td>
</tr>
<tr>
<td>$\sinh(\beta t)$</td>
<td>$\frac{\beta}{s^2 - \beta^2}$</td>
</tr>
<tr>
<td>$\text{Step}(t, A)$</td>
<td>$\frac{e^{-sA}}{s}$</td>
</tr>
<tr>
<td>$\delta(t - A)$</td>
<td>$e^{-sA}$</td>
</tr>
</tbody>
</table>
6 General Laplace-Transform Identities

This table is stuff you need to have on board your gray matter for the rest of the semester. Rote memorization would not be a bad reaction to this, though the derivations of the various identities are fairly straightforward.

<table>
<thead>
<tr>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g) )</td>
</tr>
<tr>
<td>( \mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0) )</td>
</tr>
<tr>
<td>( \mathcal{L}(e^{At}f(t))(s) = \mathcal{L}(f)(s - A) )</td>
</tr>
<tr>
<td>( \mathcal{L}(\text{Step}(t; A)f(t)) = e^{-sA}\mathcal{L}(f(t + A)) )</td>
</tr>
<tr>
<td>( e^{-sA}\mathcal{L}(f(t))(s) = \mathcal{L}(\text{Step}(t; A)f(t - A))(s) )</td>
</tr>
<tr>
<td>( \frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}((-t)f(t)) )</td>
</tr>
<tr>
<td>( \mathcal{L}(f)\mathcal{L}(g) = \mathcal{L}(f \odot g) ) , where ( (f \odot g)(t) = \int_{0}^{t} f(\tau)g(t - \tau) d\tau )</td>
</tr>
</tbody>
</table>
7 3/20/07 - Tuesday - Day 36

1 Involving simple partial-fractions review:

\[ y' + 4y = e^{8t} \quad \text{with} \quad y(0) = 7 \]

transforms into

\[ Y(s) = \frac{7}{s + 4} + \frac{1}{12} \left\{ \frac{1}{s - 8} - \frac{1}{12} \right\} \]

so that

\[ y(t) = \frac{83}{12} e^{-4t} + \frac{1}{12} e^{8t} \]

2 \( \mathcal{L}(\cos(\beta t)) \) and \( \mathcal{L}(\sin(\beta t)) \) via Euler’s formula.

3 \( \mathcal{L}(e^{At}f(t)) \ (s) = \mathcal{L}(f) \ (s - A) \) and its effects:

(i) \( e^{At}t^3 \)

(ii) \( \frac{1}{(s + 10)^3} \)

(iii) \( e^{3t} \cos(2t) \)

(iv) \( \frac{s + 8}{(s + 8)^2 + 73} = \mathcal{L}(e^{-st} \cos(t \sqrt{73})) \)

4 The initial-value problem

\[ y'' + 8y' + 15y = 64e^{-8t} \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = -8 \]

transforms to

\[ Y(s) = \frac{s}{(s + 3)(s + 5)} + \frac{64}{(s + 3)^2(s + 5)} \]
8 3/21/07 - Wednesday - Day 37

1 We continued the example from Tuesday. We expanded the two summands in the $Y(s)$ formula using partial fractions:

$$Y(s) = \frac{1}{2} \left[ \frac{5}{s+5} - \frac{3}{s+3} \right] + \left[ \frac{-16}{s+3} + \frac{32}{(s+3)^2} + \frac{16}{s+5} \right]$$

so that

$$y(t) = \frac{1}{2} \left[ 5e^{-5t} - 3e^{-3t} \right] + \left[ -16e^{-3t} + 32te^{-3t} + 16e^{5t} \right]$$

or

$$y(t) = \frac{37}{2} e^{-5t} - \frac{35}{2} e^{-3t} + 32te^{-3t}$$

2 The Laplace transform and the Step function:

(a) $L(\text{Step}(t, A)) = \frac{e^{-sA}}{s}$

(b) $L(\text{Step}(t, A)f(t)) = e^{-sA}L(f(t + A))$

(c) Let $f(t) = g(t - A)$, then $f(t + A) = g(t)$ and in the just previous formula, we have

$$L(\text{Step}(t, A)g(t - A)) = e^{-sA}L(g(t))$$

(d) An example of inverse transforming involving this last formula:

$$e^{-2\pi s} \left( \frac{2s + 3}{s^2 + 1/64} \right) = e^{-2\pi s} \left( 2 \left[ \frac{s}{s^2 + 1/64} \right] + 3 \cdot 8 \left[ \frac{1/8}{s^2 + 1/64} \right] \right)$$

$$= e^{-2\pi s} \left( 2 \cos \left( \frac{t}{8} \right) + 24 \sin \left( \frac{t}{8} \right) \right)$$

$$= L \left( \text{Step}(t, 2\pi) \left[ 2 \cos \left( \frac{t - 2\pi}{8} \right) + 24 \sin \left( \frac{t - 2\pi}{8} \right) \right] \right)$$
9  3/23/07 - Friday - Day 38

1. A side trip on \( \frac{d}{ds} \mathcal{L} (f(t)) = \mathcal{L} ((-t)f(t)) \)

2. So \( \mathcal{L} (t \cos(\beta t)) = \frac{s^2 - \beta^2}{(s^2 + \beta^2)^2} \)

3. And by PFD, \( \frac{1}{(s^2 + \beta^2)^2} = \mathcal{L} \left( \frac{\sin(t) - t \cos(t)}{2} \right) \)

4. We began with a Laplace-transform treatment of this problem (2/7/07):

   A tank initially holds 300 gallons of a solution which is 50% alcohol by volume. Throughout the problem, the inflow and outflow rates are identical: 3 gallons per minute. For the first ten minutes, the inflow is a 20% solution, but thereafter, the inflow is a 75% solution.

   Find a formula for the amount of solution in the tank in terms of \( t \) (minutes).

We translated this story into the initial-value problem

\[
y' + \frac{1}{100}y = \frac{3}{100} \left[ 20 + 55 \text{Step}(t, 10) \right] \quad \text{with} \quad y(0) = 150.
\]

Taking Laplace transforms on both sides:

\[
sY - y(0) + \frac{1}{100} Y = \frac{3}{100} \left[ \frac{20}{s} + e^{-10s} \frac{55}{s} \right]
\]

from which we got

\[
Y = \frac{150}{s + 1/100} + \frac{3}{100} \left[ \frac{20}{s(s + 1/100)} + e^{-10s} \frac{55}{s(s + 1/100)} \right]
\]

5. Lookahead: transform back to \( y(t) \).
10 4/2/07 - Monday - Day 39

1 The homework problem of finding $\mathcal{L}(f)$, where $f(t) = te^{at}\text{Step}(t, 1)$, using the general identities. Corrected Fri Apr 6 13:56:11 MDT 2007

2 The 3/23/07 initial-value problem solution:

$$y(t) = \begin{cases} 
60 + 90e^{-t/100} & \text{if } 0 \leq t \leq 10 \\
225 + (90 - 165e^{1/10})e^{-t/100} & \text{if } t > 10
\end{cases}$$

3 The power-series coefficients of the product of two power series is given by the convolution of the coefficient sequences of the respective factors.

4 Lookahead: the function whose Laplace transform is the product of two transforms is related how to the functions yielding the factors?
11 4/3/07 - Tuesday - Day 40

1 Put convolution on hold. I shouldn’t have started it up on Monday.

2 What is meant, graphically and algebraically, by the following?

\[ T > 0 \] is a period of function \( f \).

3 If \( T > 0 \) is a period of function \( f \), then

\[
\mathcal{L}(f)(s) = \left( \int_0^T f(t)e^{-st} \, dt \right) \sum_{k=0}^{\infty} (e^{-sT})^k
\]

4 We ran up against the MATH-175 facts:

\[
\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \quad \text{and} \quad \frac{1}{1+r} = \sum_{k=0}^{\infty} (-1)^k r^k
\]

5 We took on a slight generalization of Example 5.3.2: let \( 0 < a < T \), then

\[
\mathcal{L}(\text{SqWave}(t, T, a)) = \frac{1 - e^{-sa}}{s} \sum_{k=0}^{\infty} (e^{-sT})^k
\]

If we let \( a = T/2 \) as in Example 5.3.2, then

\[
\mathcal{L}(\text{SqWave}(t, T, T/2)) = \frac{1}{s} \sum_{k=0}^{\infty} (-1)^k (e^{-sT/2})^k
\]

6 Lookahead: an initial-value problem involving some sort of jagged periodic function.
Today we considered the story:

A 2000-gallon tank contains only pure water at 8 AM. Solution flows in at 40 gal/hr and out at 40 gal/hr.

For twelve hours, from 8 AM until 8 PM, the inflow is 90% alcohol. Then, from 8 PM until 8 AM, the inflow is pure water. From 8 AM, the cycle repeats itself.

Find a formula for the amount of alcohol (gallons) in the tank as a function of time (hours since the first 8 AM).

We found the alcohol inflow rate to be given by

\[ 40 \cdot 0.9 \cdot \text{SqWave}(t, 24, 12), \]

so that the story translates into the initial-value problem

\[ y'(t) + \frac{1}{50} y(t) = 36 \text{SqWave}(t, 24, 12) \quad \text{with} \quad y(0) = 0. \]

Taking Laplace transforms:

\[ sY(s) + \frac{1}{50} Y(s) = \frac{36}{s} \cdot \frac{1 - e^{-12s}}{1 - e^{-24s}} \]

or

\[ \left( s + \frac{1}{50} \right) Y(s) = \frac{36}{s} \cdot \frac{1}{1 + e^{-12s}} \]

whence

\[ Y(s) = \frac{36}{s(s + 1/50)} \cdot \frac{1}{1 + e^{-12s}} = \frac{36}{s(s + 1/50)} \cdot \left( \sum_{k=0}^{\infty} (-1)^k e^{-12sk} \right). \]

Now

\[ \frac{1}{s(s + 1/50)} = 50 \left( -s \cdot \frac{s}{s + 1/50} \right) = 50 \mathcal{L} \left( 1 - e^{-t/50} \right). \]

Thus

\[ Y(s) = 36 \cdot 50 \cdot \sum_{k=0}^{\infty} (-1)^k e^{-12ks} \mathcal{L} \left( 1 - e^{-t/50} \right) \]

\[ = 1800 \cdot \sum_{k=0}^{\infty} (-1)^k \mathcal{L} \left( \text{Step}(t, 12k) \left[ 1 - \exp(-t - 12k)/50 \right] \right) \]
so that

\[ y(t) = 1800 \sum_{k=0}^{\infty} (-1)^k \text{Step}(t, 12k) \left( 1 - e^{-t/50} e^{(6/25)k} \right). \]

For the first 12 hours, we have a learning curve:

\[ y(t) = 1800 \left( 1 - e^{-t/50} \right). \]

For the second 12 hours, we have an exponential-decay curve:

\[ y(t) = 1800 \left( e^{6/25} - 1 \right) e^{-t/50}. \]
13 Assignment #13 Grading Notes

Last Update: Thu Apr 5 10:35:42 MDT 2007

1. $e^{-As} \mathcal{L}(f) = \mathcal{L}(\text{Step}(t, A) f(t-A))$

2. Click here for a partial-fractions summary. And note especially a repeated first-degree factor in the denominator

3. If your $Y(s)$ has a part like this:

$$G(s) = \frac{\frac{e^{-4s}}{s-5} + \frac{7s}{s-5}(s+8)}{(s-5)(s+8)},$$

you may think of doing a partial-fractions decomposition. But you need to keep the $e^{-4s}$ out of the PFD process. So you want to do something like this:

$$G(s) = e^{-4s} \left( \frac{1}{s-5} - \frac{1}{s+8} \right) + \frac{7s}{(s-5)(s+8)}.$$

From here, you can subcontract out PFDs for

$$G_1(s) = \frac{1}{(s-5)(s+8)} = \frac{1}{13} \left( \frac{1}{s-5} - \frac{1}{s+8} \right) = \mathcal{L} \left( \frac{1}{13} (e^{5t} - e^{-8t}) \right)$$

(with the $e^{-4s}$ sidelined) and for

$$G_2(s) = \frac{7s}{(s-5)(s+8)} = \frac{1}{13} \left( \frac{35}{s-5} + \frac{56}{s+8} \right) = \mathcal{L} \left( \frac{1}{13} (35e^{5t} + 56e^{-8t}) \right).$$

Putting these subcontracts back into the whole enchilada:

$$G(s) = e^{-4s}G_1(s) + G_2(s)$$

$$= \mathcal{L} \left( \text{Step}(t, 4) \frac{1}{13} (e^{5(t-4)} - e^{-8(t-4)}) \right) + \mathcal{L} \left( \frac{1}{13} (35e^{5t} + 56e^{-8t}) \right).$$
14 4/6/07 - Friday - Day 42

1. A detailed walkthrough of the homework problem:

\[ y'' - 2y' + y = \text{Step}(t, 1) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0 \]

2. Refer to page 323 of the text: we went through a derivation of the convolution-product formula from the point of view that the convolution of \( f \) and \( g \) is the function that fills the blank:

\[ \mathcal{L}(f) \mathcal{L}(g) = \mathcal{L}(\___). \]
15  4/9/07 - Monday - Day 43

1  We went through some of the section-4.4 stuff on resistance, inductance, and capacitance in electric circuits.

2  Derivation of two second-order linear differential equations for a simple LRC circuit: one for charge $Q(t)$ and one for current $I(t)$.

3  We looked at a Laplace-transform solution of the $y(t) = I(t)$ initial-value problem in Example 4.4.1. This split the solution formula of the initial value problem $L(y) = f$ with $y(0) = y_0$ and $y'(0) = v_0$ into the sum of the solutions of the initial-value problems

   (i) $L(y) = 0$ with $y(0) = y_0$ and $y'(0) = v_0$

   (ii) $L(y) = f$ with $y(0) = 0$ and $y'(0) = 0$
4/10/07 - Tuesday - Day 44

1. After finishing up Monday’s agenda, we scratched the surface for Dirac’s Delta “Function”, which was hatched by physicists studying quantum mechanics in order to realize sharp blows to the system.

2. $\mathcal{L}(\delta(t - A)) = e^{-sA}$
17 4/11/07 - Wednesday - Day 45

1 Continuing from Tuesday, the initial-value problem

\[ y'' - 4y' + 4y = 3\delta(t - 1) + \delta(t - 2) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1 \]

has solution

\[ y(t) = (1 - t)e^{\frac{2}{5}t} + 3 \text{Step}(t, 1)(t - 1)e^{\frac{2}{5}(t-1)} + \text{Step}(t, 2)(t - 2)e^{\frac{2}{5}(t-2)} \]

2 We began (and continued into Friday) looking at the (two-compartment) two-tank example which was intended to motivate our matrix-algebra work:

One-hundred-gallon Tank X initially is full of a 90% alcohol solution. Also:
(i) a 20 gal/min pure-water input
(ii) a 20 gal/min solution input from Tank Y
(iii) a 20 gal/min solution output to Tank Y
(iv) a 20 gal/min solution dump into the outside world.
(v) \( x(t) \) gallons of alcohol in Tank X \( t \) minutes after start.

Two-hundred-gallon Tank Y initially is full of a 40% alcohol solution. Also:
(i) a 20 gal/min pure-water input
(ii) a 20 gal/min solution input from Tank X
(iii) a 20 gal/min solution output to Tank X
(iv) a 20 gal/min solution dump into the outside world.
(v) \( y(t) \) gallons of alcohol in Tank Y \( t \) minutes after start.

From this story we derived the first-order linear system

\[
\begin{align*}
x' &= -\frac{2}{5}x + \frac{1}{10}y \\
y' &= \frac{1}{5}x - \frac{1}{5}y.
\end{align*}
\]

A solution of this system can be thought of as parametric equations for the path of a bug crawling about in the \( xy \)-plane: his position vector is \( \mathbf{r}(t) \), given by

\[
\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},
\]

while his velocity vector, \( \mathbf{v}(t) \) is given by

\[
\mathbf{v}(t) = \mathbf{r}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}.
\]
Continuing with Wednesday’s two-tank bug, we set about to draw a rough direction field for the system.

Using the regions for $x' = 0$, $y' = 0$, $x' > 0$, $y' > 0$, $x' < 0$, and $y' < 0$, and arrived at something like this:

![Diagram showing direction field for two-tank system]

The dashed lines are to suggest solutions which are straight lines through the origin. It’s these lines that we are after via matrix theory.
2. We also started solving systems of many equations in many unknowns via the \textbf{Gauss Elimination Method}.

\section*{19 4/16/07 - Monday - Day 47}

1. Spent the class time looking at solving the assignment-#14 problems.
20 4/17/07 - Tuesday - Day 48

1 We began class with an RREF sales pitch and a list of the Three Elementary Operations. We should look upon solving systems by the Gauss Elimination Method (GEM) as a process of using the elementary operations to reduce the augmented matrix to RREF.

From there the nature of the solution set for the system is quite clear. Especially in the many-solutions case, where one is expected to write a vector solution formula as in the cross-product example below.

2 Matrifying a system of linear equations in terms of a matrix-vector product.

3 Rotation through $\theta$ in the $xy$-plane via matrix multiplication brought in from the addition formulas for $\cos(\alpha + \theta)$ and $\sin(\alpha + \theta)$.

4 The linearity of matrix-vector multiplication:

$$A(\alpha \vec{x} + \beta \vec{y}) = \alpha A \vec{x} + \beta A \vec{y}$$

As an application of this, we solved

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \times \vec{x} = \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix}.$$ 

by reducing it to the augmented matrix

$$\begin{bmatrix} 0 & -1 & -3 & 5 \\ 1 & 0 & -2 & 6 \\ 3 & 2 & 0 & 8 \end{bmatrix}$$

We found the solution

$$\vec{x} = \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$ 

We checked by showing

$$\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix}$$ and $$\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
The above example shows an analogy with the solutions of a 2OLDE:

(a) the vector \( \begin{bmatrix} 6 \\ -5 \\ 0 \end{bmatrix} \) acts like a \( y_d \) – it is a solution of the “driven” system

\[
\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 8 \end{bmatrix},
\]

(b)

(c) the vector \( \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \) acts like a \( y_h \) – it is a solution of the “undriven” system

\[
\begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
\]

21 Link to Back Issues of the Diary

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