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First Problem:

To find k so that $y(t) = e^{kt}$ is a solution of $y'' - y = 0,$ calculate the derivatives of this $y,$

$$\begin{aligned} y &= e^{kt} \\ y' &= ke^{kt} \\ y'' &= k^2 e^{kt}, \end{aligned}$$

then substitute into the differential equation and solve for k:

$$\begin{aligned} y'' - y &= 0 \\ k^2 e^{kt} - e^{kt} &= 0 \\ e^{kt}\bigl(k^2 - 1\bigr) &= 0 \\ k^2 - 1 &= 0 \quad \text{(because } e^{kt} \neq 0\text{)} \\ (k-1)(k+1) &= 0 \quad \text{(safest to factor, if possible).} \end{aligned}$$

From this we see two k values: $k = \pm 1.$

Second Problem:

Every time we "antidifferentiate", we cause another constant:

$$\begin{aligned} y''' &= 2 \\ y'' &= 2t + A \\ y' &= t^2 + At + B \\ y &= \frac{t^3}{3} + \frac{At^2}{2} + Bt + C \quad \text{(my answer)} \end{aligned}$$

One could redo the constants at this point:

$$y = \frac{t^3}{3} + C_1 t^2 + C_2 t + C_3,$$

where $C_1 = A/2,$ $C_2 = B,$ and $C_3 = C.$

Third Problem, part (a):

This problem needed verbiage. You need to write some connective stuff to show me that you *do so* know what's going on.

To show that

$$y = C_1 \sin(2t) + C_2 \cos(2t)$$

is a solution of the differential equation $y'' + 4y = 0,$ for all possible values of constants C_1 and $C_2,$ you must substitute this formula into the differential equation.

Equipment:

$$\begin{aligned} \sin'(\theta) &= \cos(\theta) \\ \cos'(\theta) &= -\sin(\theta) \quad \text{(got the "minus"?)} \end{aligned}$$

along with the **Chain Rule:**

$$\text{If} \qquad G(t) = f(\, u(t)\,) \qquad \text{then} \qquad G'(t) = f'(\, u(t)\,)u'(t).$$

This means, for instance, that

$$\frac{d}{dt}\cos(2t) = -2\sin(2t).$$

So, a solution of problem 11(a) goes like this: compute the derivatives of our given function

$$\begin{aligned} y &= C_1 \sin(2t) + C_2 \cos(2t) \\ y' &= 2C_1 \cos(2t) - 2C_2 \sin(2t) \quad \text{(got the "minus"?)} \\ y'' &= -4C_1 \sin(2t) - 4C_2 \cos(2t). \end{aligned}$$

Here are two ways to go on:

(a) Observe above that the last line says $y'' = -4y,$ or $y'' - 4y = 0,$ and we are done. This works fine for this simple differential equation.

(b) Substitute the derivatives into the left-hand side of the differential equation, and then, as in a trig-identity proof, show that you get the right-hand side. This works for more complicated differential equations, and keeps the grader off your back:

$$\begin{aligned} y'' + 4y &= \big[-4C_1 \sin(2t) - 4C_2 \cos(2t)\big] + 4\big[C_1 \sin(2t) + C_2 \cos(2t)\big] \\ &= \big(-4C_1 + 4C_1\big)\sin(2t) + \big(-4C_2 + 4C_2\big)\cos(2t) \\ &= 0. \end{aligned}$$

Either way, we have shown that the given function is a solution of the differential equation for all possible values of the constants C_1 and $C_2.$

Third Problem, part (b):

Now we must find values for the constants C_1 and C_2 in order that $y = C_1 \sin(2t) + C_2 \cos(2t)$ be a solution of the initial-value problem:

$$y'' + 4y = 0 \qquad \text{with} \qquad y(\pi/4) = 3 \qquad \text{and} \qquad y'(\pi/4) = -2$$

The initial condition $y(\pi/4) = 3,$ subsitituted into the y formula gives us

$$3 = C_1 \sin(\pi/2) + C_2 \cos(\pi/2) \qquad \text{or} \qquad C_1 = 3.$$

The initial condition $y'(\pi/4) = -2,$ subsitituted into the y' formula gives us

$$-2 = 2C_1 \cos(\pi/2) - 2C_2 \sin(\pi/2) \qquad \text{or} \qquad -2 = -2C_2 \qquad \text{or} \qquad C_2 = 1.$$

So the back-of-the-book answer to this: $C_1 = 3$ and $C_2 = 1.$

We weren't asked for the solution to the initial-value problem, but here it is:

$$y(t) = 3\sin(2t) + \cos(2t).$$

Fourth Problem:

The problem of showing that $y = 2e^{-4t}$ is a solution of the initial-value problem

$$y' + ky = 0 \qquad \text{with} \qquad y(0) = y_0$$

splits into two parts:

(a) $y_0 = y(0) = 2e^{-4(0)} = 2e^0 = 2$

(b) Finding k is like stuff we did in this assignment's First Problem: subsitute derivatives into the differential equation:

$$\begin{aligned} y' + ky &= 0 \\ \big(-4(2e^{-4t})\big) + k\big(2e^{-4t}\big) &= 0 \\ e^{-4t}\big(-8 + 2k\big) &= 0 \\ -8 + 2k &= 0 \qquad (\text{since } e^{-4t} \neq 0) \\ 2(k - 4) &= 0, \end{aligned}$$

whence $k = 4.$

So the back-of-the-book answer here is $y_0 = 2$ and $k = 4.$

Fifth Problem:

This is like part (a) of our Third Problem. Here we write down the given function and its first two derivatives, then substitute them into the differential equation $y'' - 4y = 0$ to see that the given function is a solution no matter what the values of C_1 and C_2 may be:

$$\begin{aligned} y &= C_1e^{2t} + C_2e^{-2t} \\ y' &= 2C_1e^{2t} - 2C_2e^{-2t} \\ y'' &= 4C_1e^{2t} + 4C_2e^{-2t} \end{aligned}$$

so that, substituting into the differential equations left-hand side:

$$\begin{aligned} y'' - 4y &= \left(4C_1e^{2t} + 4C_2e^{-2t}\right) - 4\left(C_1e^{2t} + C_2e^{-2t}\right) \\ &= (4C_1 - 4C_1)e^{2t} + (4C_2 - 4C_2)e^{-2t} \\ &= 0. \end{aligned}$$

This shows that $y = C_1e^{2t} + C_2e^{-2t}$ is always a solution of $y'' - 4y = 0,$ regardless of the values of C_1 and $C_2.$

Sixth Problem:

The Sixth and Seventh Problems are concerned with using the results of the Fifth Problem to get formulas for solutions to initial-value problems involving $y'' - 4y = 0.$

So, in both of these problems, we impose the given initial conditions on the formula $C_1e^{2t} + C_2e^{-2t}.$ In each case this gives us a system of two equations in the unknowns C_1 and $C_2.$

You should probably write this system down in "display form" with a box around it – part of a campaign to persuade the grader you know what's up.

Then solve the system and apply the values of C_1 and C_2 to the formula $C_1e^{2t} + C_2e^{-2t}$ to arrive at the solution of the initial-value problem.

In the Sixth Problem we have

$$\begin{aligned} y &= C_1e^{2t} + C_2e^{-2t} \\ y' &= 2C_1e^{2t} - 2C_2e^{-2t}, \end{aligned}$$

or, imposing the initial conditions

$$\begin{aligned} 2 = y(0) &= C_1e^0 + C_2e^0 \\ 0 = y'(0) &= 2C_1e^0 - 2C_2e^0, \end{aligned}$$

which gives us the system

$$\begin{aligned} C_1 + C_2 &= 2 \\ 2C_1 - 2C_2 &= 0 \end{aligned}$$

(this is the puppy you want to "display" before launching into the solution). Solution gives $C_1 = C_2 = 1$ so that the initial-value problems solution is

$$y = e^{2t} + e^{-2t}.$$

This would be the back-of-the-book solution.

Seventh and Last Problem:

Imposing the initial conditions as in the Sixth Problem:

$$\begin{aligned} 1 = y(0) &= C_1e^0 + C_2e^0 \\ 2 = y'(0) &= 2C_1e^0 - 2C_2e^0, \end{aligned}$$

which gives us the system

$$\begin{aligned} C_1 + C_2 &= 1 \\ 2C_1 - 2C_2 &= 2 \end{aligned}$$

or

$$\begin{aligned} C_1 + C_2 &= 1 \\ C_1 - C_2 &= 1 \end{aligned}$$

Solution gives $C_1 = 1$ and $C_2 = 0$ so that the initial-value problems solution is

$$y = e^{2t}.$$