M-333 #12 Grading Notes 10/16/07

First Problem For the characteristic equation: $9y''-6y'+y=0$; $9r^2-6r+1=0$; $(3r-1)^2=0$ $\therefore$ $r=\frac{1}{3}$. In view of ICs $y(3)=-2$ & $y'(3)=-\frac{5}{3}$ let

Gen Soln: $$y=e^{\frac{t-3}{3}}\left[C_1+C_2(t-3)\right]$$

Now $y'=\frac{1}{3}y+e^{(t-3)/3}\left[C_2\right]$, so that the ICs yield

$$C_1=y(3)=-2$$

$$C_2=y'(3)-\frac{1}{3}y(3)=-\frac{5}{3}-\frac{1}{3}(-2)=-\frac{3}{3}=-1$$

$\therefore$ IVP Solution:
$$y=e^{\frac{t-3}{3}}\left[-2-1(t-3)\right] \text{ or } e^{-1}e^{t/3}\left[1-t\right]$$

Second Problem $25y''+20y+4y=0$ yields $25r^2+20r+4=0$; $(5r+2)^2=0$; $r=-2/5$

Because the ICs $y(5)=4e^{-2}$ and $y'(5)=-\frac{3}{5}e^{-2}$ happen at $t=5$, we take as general solution
$$y=e^{-\frac{2}{5}(t-5)}\left[C_1+C_2(t-5)\right]$$

for which
$$y'=-\frac{2}{5}y+e^{-\frac{2}{5}(t-5)}\left[C_2\right]$$

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Imposing the ICs:

$$C_1 = y(5) = 4e^{-2}$$

$$C_2 = y'(5) + \frac{2}{5}y(5) = -\frac{3}{5}e^{-2} + \frac{8}{5}e^{-2}$$

so that the IVP solution is

$$y(t) = e^{-\frac{2}{5}(t-5)}\left[4e^{-2} + e^{-2}(t-5)\right]$$

$$\text{or } e^{-\frac{2}{5}t}\left[4 + (t-5)\right] \text{ or } e^{-2/5}(t-1)$$

<u>Third Problem</u> We are given that $y_1 = e^t$ is a solution of the 2OLDE

$$y'' - \left(2 + \frac{n-1}{t}\right)y' + \left(1 + \frac{n-1}{t}\right)y = 0,$$

where n is a positive-integer constant. Note that this is not a constant-coefficient DE, so our characteristic-equation (used with 2OLDE=0 CC) avails us naught.

The reduction-of-order shtik starts with LET $u(t)$ be a function of which we require that $y_2(t) = u(t)\,y_1(t)$

be a solution of our 2OLDE=0.

Now $y_2 = uy_1 = ue^t$, so $y_2' = (u+u')e^t$, and

$y_2'' = (u''+2u'+u)e^t$.

Multiply the 2OLDE=0 through by t (just to decrease irritation) and plug in y_2:

$$ty_2'' - (2t+n-1)y_2' + (t+n-1)y_2 = 0$$

$$t(u''+2u'+u)e^t - (2t+(n-1))(u+u')e^t + (t+(n-1))ue^t = 0$$

Divide out the e^t and collect:

$$u''\left[t\right] + u'\left[2t-(2t+(n-1))\right] + u\left[t-(2t+(n-1))+t+(n-1)\right] = 0$$

$$u''t + u'\left[-(n-1)\right] = 0$$

Letting $v = u'$, we have 1OLDE $v't - (n-1)v = 0$

or $v' + \left(-\frac{(n-1)}{t}\right)v = 0$.

$p(t) = -\frac{(n-1)}{t}$ and $P(t) = -(n-1)\ln(t)$

or $P(t) = \ln(t^{1-n})$. $\therefore IF(t) = e^{P(t)} = t^{1-n}$

$\therefore$ Multiplying in the IF yields

$$(t^{1-n}v)' = 0;\ t^{1-n}v = A;\ v = At^{n-1}$$

or $u' = At^{n-1}$ so that $u = A\frac{t^n}{n} + B$

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And so we have a solution of the 2OLDE=0:

$$y_2 = u y_1 = \left(A\frac{t^n}{n} + B\right)e^t = C_1 t^n e^t + C_2 e^t$$

Since we already have $y_1 = e^t$ as a solution, we slim y_2 down to $\underline{y_2 = t^n e^t}$

To show $\{y_1, y_2\}$ is an FSS,

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & t^n e^t \\ e^t & nt^{n-1}e^t + t^n e^t \end{vmatrix}$$

$$= e^t \begin{vmatrix} 1 & t^n e^t \\ 1 & (nt^{n-1} + t^n)e^t \end{vmatrix} = e^{2t} \begin{vmatrix} 1 & t^n \\ 1 & nt^{n-1} + t^n \end{vmatrix}$$

$= e^t n t^{n-1}$ which is non-zero on any interval not containing $t=0$.

But these are the sorts of intervals the Existence and Uniqueness Theorem guarantees solutions upon.

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Fourth Problem For $2y'' - 2y' + y = 0$ we have

$2r^2 - 2r + 1 = 0$, for which $b^2 - 4ac = -4 = (\pm 2i)^2$

$\therefore\ r = \dfrac{+2 \pm 2i}{4} = \frac{1}{2} \pm \frac{1}{2} i$

Now the ICs : $y(-\pi) = 1$ and $y'(-\pi) = -1$ suggest a general-solution form :

$$y(t) = e^{\frac{1}{2}(t+\pi)} \left[C_1 \cos\left(\frac{t+\pi}{2}\right) + C_2 \sin\left(\frac{t+\pi}{2}\right) \right]$$

for which

$$y'(t) = \frac{1}{2} y(t) + e^{\left(\frac{t+\pi}{2}\right)} \left(\frac{1}{2}\right) \left[-C_1 \sin\left(\frac{t+\pi}{2}\right) + C_2 \cos\left(\frac{t+\pi}{2}\right) \right]$$

$$C_1 \cos(0) + C_2 \sin(0) = y(-\pi) = 1 \quad \text{or } C_1 = 1$$

$$\frac{1}{2}\left[-C_1 \sin(0) + C_2 \cos(0) \right] = y'(-\pi) - \frac{1}{2} y(-\pi) = -1 - \frac{1}{2}(1) = -\frac{3}{2}$$

$$\text{or} \quad \frac{1}{2} C_2 = -\frac{3}{2} \quad \text{so} \quad C_2 = -3$$

$\therefore$ IVP solution ; $y(t) = e^{\left(\frac{t+\pi}{2}\right)} \left[\cos\left(\frac{t+\pi}{2}\right) - 3 \sin\left(\frac{t+\pi}{2}\right) \right]$

or $e^{\pi/2} e^{t/2} \left[-\sin\left(\frac{t}{2}\right) - 3 \cos\left(\frac{t}{2}\right) \right]$.

<u>Fifth Problem</u> $y''+4y'+5y=0$ yields $r^2+4r+5=0$

$(r^2+4r+4)+1=0$; $(r+2)^2=-1$; $r+2=\pm i$

$r=-2\pm i$.

Since the ICs, $y(\frac{\pi}{2})=\frac{1}{2}$ and $y'(\frac{\pi}{2})=-2$, are taken at $t=\frac{\pi}{2}$, we write the general solution:

$$y=e^{-2(t-\frac{\pi}{2})}\left[C_1\cos\left(t-\frac{\pi}{2}\right)+C_2\sin\left(t-\frac{\pi}{2}\right)\right]$$

for which

$$y'=-2y+e^{-2(t-\frac{\pi}{2})}\left[-C_1\sin\left(t-\frac{\pi}{2}\right)+C_2\cos\left(t-\frac{\pi}{2}\right)\right]$$

so that

$$C_1\cos(0)+C_2\sin(0)=y\left(\tfrac{\pi}{2}\right)=\tfrac{1}{2}$$

$$\therefore C_1=\tfrac{1}{2}$$

and

$$-C_1\sin(0)+C_2\cos(0)=y'\left(\tfrac{\pi}{2}\right)+2y\left(\tfrac{\pi}{2}\right)=-2+2\left(\tfrac{1}{2}\right)=-1$$

$$\therefore C_2=-1$$

$\therefore$ IVP solution is

$$y=e^{-2(t-\pi/2)}\left[\tfrac{1}{2}\cos\left(t-\tfrac{\pi}{2}\right)-\sin\left(t-\tfrac{\pi}{2}\right)\right]$$

$$\text{or } e^{\pi}e^{-2t}\left[\tfrac{1}{2}\sin(t)+\cos(t)\right]$$

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<u>Last Problem</u> If $t_0 = \frac{\pi}{6}$ and the IVP solution is $y = e^{t-\frac{\pi}{6}}\left[\cos(2t) - \sin(2t)\right]$, then $r = 1 \pm 2i$ and so $r - 1 = \pm 2i$ and $(r-1)^2 = (\pm 2i)^2$ or $(r-1)^2 = -4$ or $r^2 - 2r + 1 = -4$ or $r^2 - 2r + 5 = 0$ so the DE: $\underline{y'' - 2y' + 5y = 0}$

The initial conditions:

$$y_0 = y(t_0) = y\left(\tfrac{\pi}{6}\right) = e^0\left[\cos\left(\tfrac{\pi}{3}\right) - \sin\left(\tfrac{\pi}{3}\right)\right] = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$$

$$y'(t) = y(t) + e^{t-\frac{\pi}{6}}\left[-2\sin(2t) - 2\cos(2t)\right]$$

$$\therefore y_0' = y'(t_0) = y'\left(\tfrac{\pi}{6}\right) = y\left(\tfrac{\pi}{6}\right) + e^0\left[-2\sin\left(\tfrac{\pi}{3}\right) - 2\cos\left(\tfrac{\pi}{3}\right)\right]$$

$$= \frac{1-\sqrt{3}}{2} + \left[-2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{1}{2}\right)\right] = \frac{1-\sqrt{3}}{2} + \frac{-2\sqrt{3} - 2}{2}$$

$$= \frac{-1 - 3\sqrt{3}}{2}$$

$\therefore$ The IVP must be

$$y'' - 2y' + 5y = 0 \quad \text{w/} \quad y\left(\tfrac{\pi}{6}\right) = \frac{1-\sqrt{3}}{2} \;\&\; y'\left(\tfrac{\pi}{6}\right) = -\frac{1+3\sqrt{3}}{2}$$