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1 First Problem:

We are given the initial-value problem:

$$y' = \frac{1}{y^2} \qquad \text{with} \qquad y(1) = 2$$

In separating the variables in such a problem, we must remember that

(a) In this first problem, y' stands for $\dfrac{dy}{dt}$, and not for $\dfrac{dt}{dy}$.

(b) When the variables have been separated, dy and dt must **NOT** be denominator items. Think back over integrals you have known: the "differential" is always a *factor,* and never a *divisor.*

(c) Observation suggests that it's vital to write down the MATH-108-ish elementary-algebra steps:

  (i) Least Common Denominator

  (ii) Multiply Both Sides by ____

We can rewrite the differential equation as $\dfrac{dy}{dt} = y^{-2}$, and then as $y^2\,dy = dt$. The antiderivatives of both sides:

$$\frac{y^3}{3} = t + C$$

We now imppose the initial condition

$$\frac{2^3}{3} = 1 + C,$$

so that $C = \dfrac{5}{3}$. This makes the implicit solution

$$\frac{y^3}{3} = t + \frac{5}{3}.$$

The explicit solution is

$$y = \sqrt[3]{5t+3}.$$

Now, the domain of $\sqrt{x}$ is $[0, \infty)$, however, the domain of $\sqrt[3]{x}$ is $(-\infty, \infty)$, so that the validity interval for the solution of this initial-value problem is

$$(-\infty, \infty)$$

2 Second Problem:

We have the initial-value problem:

$$y' + e^y t = e^y \sin(t) \qquad \text{with} \qquad y(0) = 0$$

We rewrite the differential equation a couple of times

$$\begin{aligned} y' + e^y t &= e^y \sin(t) \\ y' &= e^y\big(\sin(t) - t\big) \\ e^{-y} y' &= \sin(t) - t \\ e^{-y}\, dy &= \big(\sin(t) - t\big)\, dt \qquad \text{separated!} \\ -e^{-y} &= -\cos(t) - \frac{t^2}{2} + C \qquad \text{antiderivatives} \end{aligned}$$

We can impose the initial condition:

$$-e^0 = -\cos(0) - 0^2/2 + C \qquad \text{or} \qquad -1 = -1 + C \qquad \text{so that} \qquad C = 0.$$

This gives the implicit solution:

$$-e^{-y} = -\cos(t) - \frac{t^2}{2}.$$

We multiply through by -1 so as to be in position to apply the famous $\ln(e^X) = X$ identity:

$$\begin{aligned} e^{-y} &= \cos(t) + \frac{t^2}{2} \\ \ln\left(e^{-y}\right) &= \ln\left(\cos(t) + \frac{t^2}{2}\right) \\ -y &= \ln\left(\cos(t) + \frac{t^2}{2}\right) \\ y &= -\ln\left(\cos(t) + \frac{t^2}{2}\right) = \ln\left(\frac{1}{\cos(t) + \frac{t^2}{2}}\right) \qquad \text{explicit.} \end{aligned}$$

For the interval, we can reason as follows:

Let $n(t) = \cos(t) + \dfrac{t^2}{2}$. Then $n(0) = 1$.

Moreover $n'(t) = -\sin(t) + t$. Now, for $t \geqslant 0$, we have

$$\sin(t) \leqslant t,$$

so that, for $t \geqslant 0$, we have $n'(t) \geqslant 0$.

Thus, for $t > 0$, $n(t)$ is “non-decreasing” as t grows.

Thus $n(t) \geqslant 1 > 0$ for $t \geqslant 0$. Noting that $n(t)$ is an even function, we have $n(t) > 0$ for all t.

Thus, for all t, we have $\cos(t) + \dfrac{t^2}{2} > 0$, so that the solution's interval of validity is $(-\infty, \infty)$.

3 Third Problem:

We are given the initial-value problem:

$$t^2\,y' + \sec(y) = 0 \qquad \text{with} \qquad y(-1) = 0.$$

Separating and antidifferentiating:

$$\begin{aligned}
t^2\,y' + \sec(y) &= 0 \\
t^2\,y' &= -\sec(y) \\
\frac{y'}{\sec(y)} &= -t^{-2} && \text{separated} \\
\cos(y)y' &= -t^{-2} \\
\sin(y) &= t^{-1} + C && \text{antiderivatives}
\end{aligned}$$

We can impose the initial condition on this last expression:

$$\sin(0) = (-1) + C \qquad \text{so that} \qquad C = 1.$$

This yields an implicit solution:

$$\sin(y) = \frac{1}{t} + 1.$$

Now, we see form the initial condition and the $\frac{1}{t}$ that we must have $t < 0.$ But also, since $-1 \leqslant \sin(\theta) \leqslant 1,$ we must have

$$-1 \leqslant \frac{1}{t} + 1 \leqslant 1.$$

Solving this

$$\begin{aligned}
-1 \leqslant &\ \frac{1}{t} + 1 \leqslant 1 \\
-2 \leqslant &\ \frac{1}{t} \leqslant 0 \\
-2t \geqslant &\ 1 \geqslant 0 && \text{because } t < 0 \\
1 &\leqslant -2t \\
-\frac{1}{2} &\geqslant t
\end{aligned}$$

This makes for an interval of validity $(-\infty, -1/2).$

For the explicit solution:

$$y = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - \arcsin\left(\frac{1}{t} + 1\right)\right) + 2\pi k,$$

where we choose the $\pm$ and the integer k so as to satisfy the initial condition $y(-1) = 0$:

$$0 = \frac{\pi}{2} \pm \left(\frac{\pi}{2} - \arcsin\left(\frac{1}{-1} + 1\right)\right) + 2\pi k$$

or

$$0 = \frac{\pi}{2} \pm \frac{\pi}{2} + 2\pi k.$$

The only way to satisfy this last equation is to choose "−" and $k = 0.$ Thus

$$y = \frac{\pi}{2} - \left(\frac{\pi}{2} - \arcsin\left(\frac{1}{t} + 1\right)\right)$$

or

$$y = \arcsin\left(\frac{1}{t} + 1\right)$$

4 Fourth Problem:

We have the initial-value problem:

$$\ln(y)y' + t = 1 \qquad \text{with} \qquad y(3) = e$$

Steps:

$$\begin{aligned} \ln(y)y' &= 1 - t \\ \ln(y)y' &= -(t-1) \\ y\big(\ln(y) - 1\big) &= -\frac{1}{2}(t-1)^2 + C \end{aligned}$$

Imposing the initial condition:

$$\begin{aligned} e\big(\ln(e) - 1\big) &= -\frac{1}{2}(3-1)^2 + C \\ e(0) &= -\frac{1}{2}(2)^2 + C \\ C &= 2 \end{aligned}$$

Therefore an implicit solution:

$$y\big(\ln(y) - 1\big) = -\frac{1}{2}(t-1)^2 + 2$$

or

$$y\ln(y/e) = -\frac{1}{2}(t-1)^2 + 2.$$

There's just no way to solve this equation for $y,$ so we aren't required to find an "interval".

5 Fifth Problem:

If $\boldsymbol{y(t)}$ is the amount of stuff present $\boldsymbol{t}$ minutes after start, then we have an initial-value problem,

$$\boldsymbol{y' = k\,y^2} \qquad \text{with} \qquad \boldsymbol{y(10) = \frac{2}{5}y(0).}$$

Solving the differential equation:

$$\begin{aligned} y' &= k\,y^2 \\ y^{-2}y' &= k \\ -y^{-1} &= kt + C \\ y^{-1} &= C - kt \qquad\qquad \text{(new } C\text{)} \\ y &= \frac{1}{C - kt} \end{aligned}$$

Now we impose the initial condition, $\boldsymbol{y(10) = \frac{2}{5}y(0)}$ **:**

$$\begin{aligned} y(10) &= \frac{2}{5}y(0) \\ \frac{1}{C - 10k} &= \frac{2}{5}\left(\frac{1}{C}\right) \\ 2C - 20k &= 5C \\ C &= -\frac{20k}{3} \end{aligned}$$

Therefore

$$y(t) = \frac{1}{-\frac{20k}{3} - kt} = \left(-\frac{1}{k}\right)\frac{1}{t + \frac{20}{3}}$$

Now we use this last formula to find $\boldsymbol{t}$ so that $\boldsymbol{y(t) = \frac{1}{4}y(0).}$

$$\begin{aligned} \left(-\frac{1}{k}\right)\frac{1}{t + \frac{20}{3}} &= \frac{1}{4}\left(-\frac{1}{k}\left(\frac{1}{0 + \frac{20}{3}}\right)\right) \\ \frac{1}{t + \frac{20}{3}} &= \frac{1}{\left(\frac{80}{3}\right)} \\ t &= 20. \end{aligned}$$

So there will be **25%** of the initial amount remaining after 20 minutes.

6 Last Problem:

Note that $\boldsymbol{y' = -y^2}$ solutions must always decrease, because $\boldsymbol{y' < 0}$. Graph C is the only graph showing a decreasing function.

Graph B appears to have two equilibria. $\boldsymbol{y' = y(4 - y)}$ has two equilibria.

$\boldsymbol{y' = y^3}$ has a zero equilibrium and has increasing solutions in the upper half-plane: Graph A.

7 Scoring:

Here's how the point values were assigned:

1 Separate-the-Variables initial-value problemwith "interval"

- (i) Solve the differential equation and impose the initial condition: 20
- (ii) Interval: 5

2 Separate-the-Variables initial-value problemwith "interval"

- (i) Solve the differential equation and impose the initial condition: 20
- (ii) Interval: 10

3 Separate-the-Variables initial-value problemwith "interval"

- (i) Solve the differential equation and impose the initial condition: 20
- (ii) Interval: 10

4 Separate-the-Variables initial-value problemwith "interval"

- (i) Solve the differential equation and impose the initial condition: 20
- (ii) No interval required.

5 Non-exponential goo: 30

6 Match graphs and differential equations: 30

For a total of 165 points.