First Problem We have $my'' + ky = 20\cos(8\pi t)$ with $y(0)=0$, $y'(0)=0$ with solution $y = \frac{1}{10}\sin(\pi t)\sin(7\pi t)$

In class we solved $y'' + \omega_0^2 y = F\cos(\omega_1 t)$ with $y(0)=0=y'(0)$:

$$y(t) = \frac{2F}{\omega_1^2-\omega_0^2}\sin\left(\frac{\omega_1+\omega_0}{2}t\right)\sin\left(\frac{\omega_1-\omega_0}{2}t\right)$$

We see that $\underline{\underline{\omega_1 = 8\pi}}$ and $\frac{8\pi+\omega_0}{2} = 7\pi$, so $\underline{\underline{\omega_0 = 6\pi}}$.

Furthermore

$$\frac{2F}{\omega_1^2-\omega_0^2} = \frac{1}{10} \text{ or } \frac{2F}{(8\pi)^2-(6\pi)^2} = \frac{1}{10} \text{ so } F = \frac{28\pi^2}{20} = \frac{7\pi^2}{5}$$

Now we must put the problem's given DE into the form $y'' + \omega_0^2 y = F\cos(\omega_1 t)$:

$y'' + \frac{k}{m}y = \frac{20}{m}\cos(8\pi t)$ so that

$$F = \frac{20}{m} \text{ or } m = \frac{20}{F} = \frac{20}{\left(\frac{7\pi^2}{5}\right)} = \underline{\underline{\frac{100}{7\pi^2}}}\text{ kg}$$

and $\frac{k}{m} = \omega_0^2$ so $k = m\omega_0^2 = \frac{100}{7\pi^2}36\pi^2 = \underline{\underline{\frac{3600}{7}}}$ N/m.

$\therefore\ m = \frac{100}{7\pi^2}$ and $k = \frac{3600}{7}$.

M-333 #15 Grading Notes.

Second Problem $my'' + \gamma y' + ky = F(t)$ with $y(0)=0$, $y'(0)=0$ with $m = 2$ kg $\gamma = 8$ kg/sec $k = 80$ N/m and $F(t) = 20e^{-t}$

(a) $2y'' + 8y' + 80y = 20e^{-t}$ or $y'' + 4y' + 40y = 10e^{-t}$

Characteristic Roots $r^2 + 4r + 40 = 0$

M-143 Solution: $r^2 + 4r + 4 = -40 + 4$

$(r+2)^2 = -36$; $r + 2 = \pm 6i$ or $r = -2 \pm 6i$

$\therefore$ The general solution of $y'' + 4y' + 40y = 0$ is

$$y_c(t) = e^{-2t}\left[C_1 \cos(6t) + C_2 \sin(6t)\right]$$

Particular solution of $y'' + 4y' + 40y = 10e^{-t}$

Guess $y_p = Ae^{-t}$ To evaluate A, substitute:

$$y_p'' + 4y_p' + 40y_p = 10e^{-t}$$

$$Ae^{-t} - 4Ae^{-t} + 40Ae^{-t} = 10e^{-t}$$

$$37A = 10 \quad \text{so} \quad A = \frac{10}{37}$$

$$\therefore y_p = \frac{10}{37}e^{-t}$$

$\therefore$ General solution of $y'' + 4y' + 40y = 10e^{-t}$ is $y(t) = y_c(t) + \frac{10}{37}e^{-t}$ aka

$$y = e^{-2t}\left(C_1 \cos(6t) + C_2 \sin(6t)\right) + \frac{10}{37}e^{-t}$$

Now to fit this with the ICs $\longrightarrow$

M-333 #15 Grading Notes. 3/7

Second Problem Continues Fitting

$y(0)=0=y'(0)$ to $y(t)=y_c(t)+\frac{10}{37}e^{-t}$, i.e.

$$y(t)=e^{-2t}\left(C_1\cos(6t)+C_2\sin(6t)\right)+\frac{10}{37}e^{-t}$$

Note that $y_c(0)=C_1$ and so

$$0=y(0)=y_c(0)+\frac{10}{37}e^0 \text{ or } 0=C_1+\frac{10}{37}$$

so $C_1=-\frac{10}{37}$ and $y_c(0)=-\frac{10}{37}$

Then $y'(t)=-2y_c(t)+e^{-2t}(6)\left[-C_1\sin(6t)+C_2\cos(6t)\right]-\frac{10}{37}$

so that

$$0=y'(0)=-2y_c(0)+6(C_2)-\frac{10}{37}$$

$$0=-2\left(-\frac{10}{37}\right)+6(C_2)-\frac{10}{37}$$

$$0=\frac{10}{37}+6C_2 \text{ or } C_2=-\frac{10}{6*37}=-\frac{5}{3*37}$$

$$\text{so } C_1=\frac{-30}{111} \text{ \& } C_2=\frac{-5}{111}$$

$\therefore$ IVP solution is

$$y(t)=-\frac{e^{-2t}}{111}\left(30\cos(6t)+5\sin(6t)\right)+\frac{10}{37}e^{-t}$$

Note that $|y(t)|\le\frac{e^{-2t}}{111}(30+5)+\frac{10}{37}e^{-t}\longrightarrow 0$ as $t\to+\infty$.

M-333 #15 Grading Notes

Third Problem Is the same as the second one except that

$F(t) = \begin{cases} 20 & 0 \le t \le \frac{\pi}{2} \\ 0 & t > \frac{\pi}{2} \end{cases}$ so DE is $y'' + 4y' + 40y = \begin{cases} 10 \text{ on } [0, \frac{\pi}{2}] \\ 0 \text{ on } (\frac{\pi}{2}, +\infty) \end{cases}$

We inherit $y_c(t) = e^{-2t}\left(C_1 \cos(6t) + C_2 \sin(6t)\right)$ although the C_i in $[0, \frac{\pi}{2}]$ turn out to differ from the C_i in $(\frac{\pi}{2}, +\infty)$. Also $y_c(0) = C_1$

$[0, \frac{\pi}{2}]$ part Find y_p for $y'' + 4y' + 40y = 10$:

eyeball: $y_p = \frac{1}{4}$

$\therefore$ General solution on $[0, \frac{\pi}{2}]$: $y(t) = y_c(t) + \frac{1}{4}$

$0 = y(0) = y_c(0) + \frac{1}{4}$; $0 = C_1 + \frac{1}{4}$ so $C_1 = -\frac{1}{4}$

so $y_c(0) = C_1 = -\frac{1}{4}$ and furthermore

$y'(t) = -2y_c(t) + 6e^{-2t}\left(-C_1 \sin(6t) + C_2 \cos(6t)\right) + 0$

$0 = y'(0) = -2\left(-\frac{1}{4}\right) + 6C_2$ or $0 = \frac{1}{2} + 6C_2$ so

$C_2 = -\frac{1}{12}$

$\therefore$ On $[0, \frac{\pi}{2}]$ we have

$$y(t) = \frac{1}{4} + \frac{e^{-2t}}{12}\left[-3\cos(6t) - \sin(6t)\right]$$

We must use this $[0, \frac{\pi}{2}]$ solution to inform the $(\frac{\pi}{2}, +\infty)$ IVP $\longrightarrow$

M-333 #15 Grading Notes

Third Problem Continues into $\left(\frac{\pi}{2}, +\infty\right)$

On $[0, \frac{\pi}{2}]$ we have

$$y(t) = \frac{1}{4} + \frac{e^{-2t}}{12}\left[-3\cos(6t) - \sin(6t)\right]$$

and

$$y'(t) = \frac{-2e^{-2t}}{12}\left[-3\cos(6t) - \sin(6t)\right] + \frac{e^{-2t}}{12}\left[18\sin(6t) - 6\cos(6t)\right]$$

so, when $t = \frac{\pi}{2}$, we have $6t = 3\pi$ and

$$\cos(6t) = \cos(3\pi) = -1$$

$$\sin(6t) = \sin(3\pi) = 0$$

$$\therefore\ y\left(\tfrac{\pi}{2}\right) = \frac{1}{4} + \frac{e^{-\pi}}{12}\left[-3(-1)\right] = \frac{1}{4} + \frac{e^{-\pi}}{4}$$

$$y'\left(\tfrac{\pi}{2}\right) = \frac{-2}{12}e^{-\pi}\left[-3(-1)\right] + \frac{e^{-\pi}}{12}\left[-6(-1)\right] = \frac{e^{-\pi}}{12}\left[-6+6\right] = 0$$

We use these values for ICs for the $\left(\frac{\pi}{2}, 0\right)$ part:

$$y'' + 4y' + 40y = 0 \quad \text{with } y(0) = \frac{1+e^{-\pi}}{4},\ y'(0) = 0.$$

Now the general solution is

$$y(t) = e^{-2t}\left(C_1\cos(6t) + C_2\sin(6t)\right).$$

We fit the ICs $\longrightarrow$

for the $\left(\frac{\pi}{2}, +\infty\right)$ part.

M-333 #15 Grading Notes (6/7)

Third Problem, $(\frac{\pi}{2}, +\infty)$ part, continues:

$$y(t) = e^{-2t}\left(C_1\cos(6t) + C_2\sin(6t)\right)$$

$$y'(t) = -2y(t) + 6e^{-2t}\left(-C_1\sin(6t) + C_2\cos(6t)\right)$$

$$\frac{1+e^{-\pi}}{4} = y(\tfrac{\pi}{2}) = e^{-\pi}\left(C_1(-1)\right) \text{ so } \frac{1+e^{-\pi}}{4} = -e^{-\pi}C_1$$

$$\text{so } C_1 = -\frac{e^{\pi}+1}{4}$$

$$0 = y'(\tfrac{\pi}{2}) = -2y(0) + 6e^{-\pi}\left(C_2(-1)\right)$$

$$0 = -2\left(\frac{1+e^{-\pi}}{4}\right) - 6e^{-\pi}C_2$$

$$0 = -2\left(\frac{e^{\pi}+1}{4}\right) - 6C_2$$

$$6C_2 = -\frac{1}{2}(e^{\pi}+1) \text{ so } C_2 = -\frac{e^{\pi}+1}{12}$$

$\therefore$ for $(\frac{\pi}{2}, \infty)$ we have

$$y(t) = \frac{e^{-2t}(e^{\pi}+1)}{12}\left(-3\cos(6t) - \sin(6t)\right)$$

for which $|y(t)| \le \frac{e^{\pi}+1}{12}(3+1)e^{-2t} \to 0$ as $t \to +\infty$.

$$y(t) = \begin{cases} \frac{1}{4} + \frac{e^{-2t}}{12}\left[-3\cos(6t) - \sin(6t)\right] & \text{for } 0 \le t \le \frac{\pi}{2} \\ (e^{\pi}+1)\,\frac{e^{-2t}}{12}\left(-3\cos(6t) - \sin(6t)\right) & \text{for } t > \frac{\pi}{2} \end{cases}$$

End of Third Problem

M-333 #15 Grading Notes.

Fourth & Last Problem

Given IVP $t^2y''' + ty'' - y' = 0$ $(t<0)$

with $y(-1) = 1$ $y'(-1) = -1$ $y''(-1) = -1$.

and functions $y_1(t) = 1$ $y_2(t) = \ln(-t)$ $y_3(t) = t^2$

(a) Check that $\{y_1, y_2, y_3\}$ is an FSS.

$y_1(t) \equiv 1$ is obviously a solution

$y_2(t) = \ln(-t)$ is a mite tricky

$$y_2'(t) = \frac{1}{(-t)} * \frac{d}{dt}(-t) \quad \text{Chain Rule!}$$

$$= \left(-\frac{1}{t}\right)(-1) = +\frac{1}{t} = t^{-1}$$

$y_2''(t) = -t^{-2}$ and $y_2''' = 2t^{-3}$

so that

$$t^2y_2''' + ty_2'' - y_2' = t^2(2t^{-3}) + t(-t^{-2}) - (t^{-1})$$

$$= 2t^{-1} - t^{-1} - t^{-1} = 0. \quad \therefore y_2 \text{ is a soln}$$

$y_3(t) = t^2$ $t^2y_3''' + ty_3'' - y_3' = t^2(0) + t(2) - (2t) = 0$

so y_3 is a solution.

ABEL'S THEOREM says a Wronskian check in $(-\infty, 0)$ is enought

$$W(-1) = \begin{vmatrix} y_1(-1) & y_2(-1) & y_3(-1) \\ y_1'(-1) & y_2'(-1) & y_3'(-1) \\ y_1''(-1) & y_2''(-1) & y_3''(-1) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & 2 \end{vmatrix} = (1)\begin{vmatrix} -1 & -2 \\ -1 & 2 \end{vmatrix} = -4$$

$\therefore \{y_1, y_2, y_3\}$ is FSS