M-333 #18 Grading Notes 11/27/07

Problem	1st	2nd	3rd(a)	3rd(b)	3rd(c)	TOTAL
Points	25	25	25	25	20	120

1st Problem $\vec{y}' = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{y}$ w/ $\vec{y}(0) = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-3)(\lambda-1)$$

Eigenvalues $\lambda=1, \lambda=2, \lambda=3$

$\lambda=1$ eigenvalue $\left[A-(1)I \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \to \left[\begin{matrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \to \left[\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda=2$ The eigenpair $\left(2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ is eyeball-able.

$\lambda=3$ $[A-(3)I \,|\, \vec{0}] \to \left[\begin{matrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

We show $\left\{ \vec{u}_1 = e^{t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_2 = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_3 = e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

is a FSS by evaluating the Wronskian at a convenient t value:

$$W(0) = \begin{vmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\therefore$ General solution is

$$\vec{y}(t) = c_1 e^{t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Impose IC on this general-solution formula

$$\vec{y}(0) = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \text{ or } c_1 e^{0}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{0}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{0}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} \text{ or } \begin{array}{cccc} c_1 & c_2 & c_3 & \\ \hline \end{array}\left[\begin{matrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} 4 \\ 3 \\ -1 \end{matrix}\right]$$

More $\longrightarrow$

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1st Problem Continues

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 4 \\ -1 & 0 & 1 & 3 \\ 1 & 0 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} c_1 = -1 \\ c_2 = 1 \\ c_3 = 2 \end{array}$$

$\therefore$ IVP solution: $y(t) = (-1)e^{t}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ or

$$= \begin{bmatrix} e^{t} + e^{2t} + 2e^{3t} \\ e^{t} \quad + 2e^{3t} \\ -e^{t} \quad + 2e^{3t} \end{bmatrix}$$

2nd Problem

$\vec{y}\,' = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 8 & 15 \\ 0 & -6 & -11 \end{bmatrix}\vec{y} \quad \vec{y}(0) = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$

Note eyeball $\left(3, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 8-\lambda & 15 \\ 0 & -6 & -11-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 8-\lambda & 15 \\ -6 & -11-\lambda \end{vmatrix} = (3-\lambda)\left[(\lambda-8)(\lambda+11) + 90\right]$$

$$= (3-\lambda)\left[\lambda^2 + 3\lambda - 88 + 90\right] = (3-\lambda)\left[\lambda^2 + 3\lambda + 2\right] = (3-\lambda)(\lambda+1)(\lambda+2)$$

$\therefore$ eigenvalues $\lambda = -2, -1, +3$.

$\lambda = -1$ eigenvector $\left[A - (-1)I \,\middle|\, \vec{0}\right] = \left[\begin{array}{ccc|c} 4 & 1 & 2 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & -6 & -10 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/12 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

yields eigenvector $\begin{bmatrix} -1/12 \\ -5/3 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 20 \\ -12 \end{bmatrix}$

$\lambda = -2$ eigenvector $\left[A - (-2)I \,\middle|\, \vec{0}\right] = \left[\begin{array}{ccc|c} 5 & 1 & 2 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/10 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

yields $\begin{bmatrix} -1/10 \\ -3/2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 15 \\ -10 \end{bmatrix}$

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2nd problem continues We check that

$$\vec{u}_1 = e^{3t}\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \vec{u}_2 = e^{-t}\begin{bmatrix}1\\20\\-12\end{bmatrix} \quad \vec{u}_3 = e^{-2t}\begin{bmatrix}1\\15\\-10\end{bmatrix}$$

constitute a FSS

$$W(0) = \left| \vec{u}_1(0) \,\vdots\, \vec{u}_2(0) \,\vdots\, \vec{u}_3(0) \right| = \begin{vmatrix}1 & 1 & 1\\0 & 20 & 15\\0 & -12 & -10\end{vmatrix} = \begin{vmatrix}20 & 15\\-12 & -10\end{vmatrix} = -200+180 \neq 0$$

$\therefore$ General Solution:

$$\vec{y}(t) = c_1 e^{3t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\20\\-12\end{bmatrix} + c_3 e^{-2t}\begin{bmatrix}1\\15\\-10\end{bmatrix}$$

Imposing the IC:

$$\vec{y}(0) = \begin{bmatrix}2\\5\\-2\end{bmatrix} \text{ or } \sum_{i=1}^{3} c_i \vec{u}_i(0) = \begin{bmatrix}2\\5\\-2\end{bmatrix} \text{ or } c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\20\\-12\end{bmatrix} + c_3\begin{bmatrix}1\\15\\-10\end{bmatrix} = \begin{bmatrix}2\\5\\-2\end{bmatrix}$$

$$\text{or } \begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & \end{matrix} \\ \left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\0 & 20 & 15 & 5\\0 & -12 & -10 & -2\end{array}\right] \end{array} \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{array}\right]$$

$\therefore$ IVP solution $\vec{y}(t) = 2e^{3t}\begin{bmatrix}1\\0\\0\end{bmatrix} + e^{-t}\begin{bmatrix}1\\20\\-12\end{bmatrix} - e^{-2t}\begin{bmatrix}1\\15\\-10\end{bmatrix}$

$$= \begin{bmatrix}2e^{3t} + e^{-t} - e^{-2t}\\ 20e^{-t} - 15e^{-2t}\\ -12e^{-t} + 10e^{-2t}\end{bmatrix}$$

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Third Problem

Part (a) Note that each tank drains via 3 pipes, and receives salt from both other tanks

$$Q_1' = -3r\frac{Q_1}{V} + \frac{rQ_2}{V} + \frac{rQ_3}{V}$$

$$Q_2' = \frac{rQ_1}{V} - 3\frac{rQ_2}{V} + \frac{rQ_3}{V}$$

$$Q_3' = \frac{rQ_1}{V} + \frac{rQ_2}{V} - 3r\frac{Q_3}{V}$$

or $$\vec{Q}' = \frac{r}{V}\begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}\vec{Q} \quad \text{w/} \quad \vec{Q}(0) = Q_0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Part (b) We find the eigenstuff for $B = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}$ and then get the "right" eigenstuff.

(top row)

$$|B - \lambda I| = \begin{vmatrix} -3-\lambda & 1 & 1 \\ 1 & -3-\lambda & 1 \\ 1 & 1 & -3-\lambda \end{vmatrix} = (-3-\lambda)\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -3-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -3-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (-3-\lambda)\left[(\lambda+3)^2 - 1\right] + 2\begin{vmatrix} 1 & -3-\lambda \\ 1 & 1 \end{vmatrix} = (-3-\lambda)\left[(\lambda+3)-1\right]\left[(\lambda+3)+1\right] + 2(\lambda+4)$$

$$= (-3-\lambda)(\lambda+2)(\lambda+4) + 2(\lambda+4) = (\lambda+4)\left[(-3-\lambda)(\lambda+2) + 2\right]$$

$$= (\lambda+4)\left[-\lambda^2 - 5\lambda - 6 + 2\right] = -(\lambda+4)\left[\lambda^2 + 5\lambda + 4\right]$$

$$= -(\lambda+4)(\lambda+4)(\lambda+1)$$

$\therefore$ B eigenvalues $\lambda = -4 \quad \lambda = -1$

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Third (z) continues

$\underline{\lambda=-4}$ $\left[B-(-4)I \mid \vec{0}\right]=\left[\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]\Rightarrow\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\underline{\lambda=-1}$ $\left[B-(-1)I \mid \vec{0}\right]=\left[\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right]\rightarrow\rightarrow\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]\Rightarrow\begin{bmatrix}1\\1\\1\end{bmatrix}$

Thus the $B=\begin{bmatrix}-3&1&1\\1&-3&1\\1&1&-3\end{bmatrix}$ eigenstuff:

$$\left(-4, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right) \quad \left(-4, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right) \quad \left(-1, \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

and so the $\frac{r}{V}\begin{bmatrix}-3&1&1\\1&-3&1\\1&1&-3\end{bmatrix}$ eigenstuff:

$$\left(-\frac{4r}{V}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right) \quad \left(-\frac{4r}{V}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right) \quad \left(-\frac{r}{V}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

We check that

$$\vec{u}_1(t)=e^{-4rt/V}\begin{bmatrix}1\\-1\\0\end{bmatrix} \quad \vec{u}_2(t)=e^{-4rt/V}\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad \vec{u}_3(t)=e^{-\frac{rt}{V}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

is FSS:

$$W(0)=\left|\vec{u}_1(0) \;\vdots\; \vec{u}_2(0) \;\vdots\; \vec{u}_3(0)\right|=\begin{vmatrix}1&1&1\\-1&0&1\\0&-1&1\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&2\\0&-1&1\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&2\\0&0&3\end{vmatrix}=3$$

So a general-solution formula:

$$Q(t)=e^{-4rt/V}\left(c_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)+c_3e^{-rt/V}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

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<u>Third Problem</u> Part (c): Impose ICs on the general solution

$\vec{Q}(0) = Q_0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We do $\vec{Q}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ first:

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 3 \end{array}\right] \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} c_1 = 0 \\ c_2 = -1 \\ c_3 = 2 \end{array}$$

$$\therefore \vec{Q}(t) = Q_0 \left\{ -e^{-\frac{4rt}{V}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2e^{-rt/V} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$