On Tuesday we were in the midst of an example: solve the initial-value problem

$$\bar{x}' = A\bar{x} + F(t) \quad \text{with} \quad \bar{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix} \quad \text{and} \quad F(t) = 4e^t \cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We begin by finding a real-valued Basic Solution Set for the undriven system $\bar{x}' = A\bar{x}$.

The characteristic polynomial:

$$\begin{vmatrix} 4 - \lambda & 5 \\ -2 & -2 - \lambda \end{vmatrix} = (\lambda - 4)(\lambda + 2) + 10$$

$$= (\lambda - 1 - 3)(\lambda - 1 + 3) + 10$$

$$= (\lambda - 1)^2 - 9 + 10 = (\lambda - 1)^2 + 1,$$

so that the eigenvalues are given by $\lambda = 1 \pm i$.

To find an eigenvector for eigenvalue $\lambda = 1 + i$:

$$\left. \begin{bmatrix} 4 - \lambda & 5 \\ -2 & -2 - \lambda \end{bmatrix} \right|_{\lambda = 1+i} = \begin{bmatrix} 4 - (1 + i) & 5 \\ -2 & -2 - (1 + i) \end{bmatrix} = \begin{bmatrix} 3 - i & 5 \\ -2 & -3 - i \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & (3 + i)/2 \\ 0 & 0 \end{bmatrix}$$

This last RREF matrix tells us that an eigenvector of form $\begin{bmatrix} a \\ b \end{bmatrix}$ must obey

$$a + \frac{3 + i}{2} b = 0,$$

so that the eigenvector we need could be one of

$$\begin{bmatrix} -(3 + i)/2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -3 - i \\ 2 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} -5 \\ 3 - i \end{bmatrix}.$$

Let’s settle on the middle one of the three.

Here is a solution of $\bar{x}' = A\bar{x}$:

$$e^{(1+i)t} \begin{bmatrix} -3 - i \\ 2 \end{bmatrix}.$$
We use the celebrated Euler’s Formula,

\[ e^{i\beta} = \cos(\beta t) + i \sin(\beta t), \]

to expand the above solution:

\[
e^{(1+i)t} \begin{bmatrix} -3 - i \\ 2 \end{bmatrix} = e^t \begin{pmatrix} \cos(t) + i \sin(t) \end{pmatrix} \begin{bmatrix} -3 - i \\ 2 \end{bmatrix}
\]

\[
= e^t \left\{ \begin{bmatrix} -3 \cos(t) + \sin(t) \\ 2 \cos(t) \end{bmatrix} + i \begin{bmatrix} -\cos(t) - 3 \sin(t) \\ 2 \sin(t) \end{bmatrix} \right\}
\]

This brings us to a real-valued general solution for the undriven \( \ddot{x}' = A\dot{x} \):

\[
\ddot{x}(t) = e^t \left\{ C_1 \begin{bmatrix} -3 \cos(t) + \sin(t) \\ 2 \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} -\cos(t) - 3 \sin(t) \\ 2 \sin(t) \end{bmatrix} \right\}
\]

6 We use the Basic Solution Set implicit in the above general-solution formula to cook up a solution matrix:

\[
X(t) = e^t \begin{bmatrix} -3 \cos(t) + \sin(t) & -\cos(t) - 3 \sin(t) \\ 2 \cos(t) & 2 \sin(t) \end{bmatrix}.
\]

We can repackage our usual IVP shtik as follows: to solve the IVP

\[
\ddot{x}' = A\ddot{x} \quad \text{with} \quad \ddot{x}(0) = \ddot{b},
\]

we use \( \ddot{x}(t) = X(t)\dddot{x} \), where \( X(0)\dddot{x} = \dddot{b} \).

7 The above IVP shtik can be streamlined if we have the “right” solution matrix. This “right” solution matrix is the one known as \( e^{At} \). Its rightness stems from the fact that the solution of the above IVP is given directly by \( \ddot{x}(t) = e^{At}\dddot{b} \). Monday, we saw how this implies that

\[
e^{At} = X(t)X(0)^{-1}
\]

For us,

\[
X(0) = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad X(0)^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}
\]

so that we have

\[
e^{At} = X(t)X(0)^{-1} = \frac{e^t}{2} \begin{bmatrix} 2 \cos(t) + 6 \sin(t) & 10 \sin(t) \\ -4 \sin(t) & 2 \cos(t) - 6 \sin(t) \end{bmatrix}
\]
The Variation of Parameters method consists of positing a solution of form

\[ \bar{y}(t) = e^{At} \bar{c}(t) \]

for the driven system

\[ \ddot{x} = A\dot{x} + F(t). \]

We saw that this implies we must have

\[ \bar{c}(t) = (e^{At})^{-1} F(t), \]

that is

\[
\begin{align*}
\bar{c}'(t) &= \frac{e^{-t}}{2} \begin{bmatrix} 2 \cos(t) - 6 \sin(t) & -10 \sin(t) \\ 4 \sin(t) & 2 \cos(t) + 6 \sin(t) \end{bmatrix} \begin{bmatrix} 4e^t \cos(t) \\ 1 \end{bmatrix} \\
&= 2 \cos(t) \begin{bmatrix} 2 \cos(t) - 6 \sin(t) \\ 4 \sin(t) \end{bmatrix}
\end{align*}
\]