The problem was to solve the initial-value problem:

$$\boldsymbol{L(y) = 8t^2 + 12e^{-t} \qquad y(0) = 0,\ y'(0) = 2,}$$

where the linear operator $\boldsymbol{L}$ is given by $\boldsymbol{L(y) = y'' - 3y' + 2y}$. See this worksheet image for a Maple solution. Here we present a by-hand solution.

(A) First we get a general-solution formula for the associated undriven equation $\boldsymbol{L(y) = 0}$

- (i) The characteristic polynomial: $\boldsymbol{m^2 - 3m + 2 = (m-2)(m-1)}$.
- (ii) The characteristic roots: $\boldsymbol{m = 2,\ m = 1}$.
- (iii) A Basic Set of Solutions: $\boldsymbol{\left\{e^{2t},\ e^{t}\right\}}$
- (iv) A general-solution formula: $\boldsymbol{y_u = C_1 e^{2t} + C_2 e^{t}}$

This shows us that the right-hand-side functions in the driven equations are not solutions of the undriven equation. This eases our lucky guessing.

(B) Next we subcontract out the particular solution $\boldsymbol{y_q}$ for $\boldsymbol{L(y) = e^{-t}}$:

- (i) We guess that $\boldsymbol{y_q = Ae^{-t}}$.
- (ii) Then $\boldsymbol{L(y_q) = Ae^{-t} + 3Ae^{-t} + 2Ae^{-t} = 6Ae^{-t}}$.
- (iii) Hence $\boldsymbol{L(y_q) = e^{-t}}$ becomes $\boldsymbol{6Ae^{-t} = e^{-t}}$ so that $\boldsymbol{A = 1/6}$.

Hence, for $\boldsymbol{L(y) = e^{-t}}$ we have $\boldsymbol{y_q = \dfrac{e^{-t}}{6}}$

(C) Next we subcontract out the particular solution $\boldsymbol{y_r}$ for $\boldsymbol{L(y) = t^2}$:

- (i) We guess that $\boldsymbol{y_r = At^2 + Bt + C}$, the most general degree-two polynomial.
- (ii) Then

$$\begin{aligned} \boldsymbol{L(y_r)} &= \boldsymbol{2A - 3(2At + B) + 2(At^2 + Bt + C)} \\ &= \boldsymbol{t^2\big[2A\big] + t\big[-6A + 2B\big] + \big[2A - 3B + 2C\big]} \end{aligned}$$

- (iii) Equating this last to the right-hand side of $\boldsymbol{L(y) = t^2}$, we get the system:

$$\begin{aligned} \boldsymbol{2A} &= \boldsymbol{1} \\ \boldsymbol{-6A + 2B} &= \boldsymbol{0} \\ \boldsymbol{2A - 3B + 2C} &= \boldsymbol{0} \end{aligned}$$

The first equation gives us $\boldsymbol{A = \dfrac{1}{2}}$, which, fed into the second equation gives $\boldsymbol{B = \dfrac{3}{2}}$, which, applied to the third equation, yields $\boldsymbol{C = \dfrac{7}{4}}$.

Hence, for $L(y) = t^2$, we have

$$y_r = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{7}{4}.$$

(D) Now we get back to a particular solution to the original equation $L(y) = 8t^2 + 12e^{-t}$ by letting

$$y_p = 8y_r + 12y_q.$$

Then, using the linearity of the operator L, we have

$$L(y_p) = L\big(8y_r + 12y_q\big) = 8L(y_r) + 12L(y_q) = 8t^2 + 12e^{-t}.$$

so that y_p is a particular solution of $L(y) = 8t^2 + 12e^{-t}$:

$$y_p = 8(y_r) + 12(y_q) = 8\left(\frac{1}{2}t^2 + \frac{3}{2}t + \frac{7}{4}\right) + 12\left(\frac{e^{-t}}{6}\right) = 4t^2 + 12t + 14 + 2e^{-t}$$

(E) We assemble a general-solution formula for the driven equation from the undriven-equation solution and the y_p:

$$y = y_p + y_u = 4t^2 + 12t + 14 + 2e^{-t} + C_1e^{2t} + C_2e^t$$

(F) Finally we impose the initial conditions on the driven equation general-solution formula.

(i) Differentiating the general-solution formula:

$$y' = 8t + 12 - 2e^{-t} + 2C_1e^{2t} + C_2e^t.$$

(ii) Substituting

$$\begin{aligned} 0 &= y(0) = 14 + 2 + C_1 + C_2 \\ 2 &= y'(0) = 12 - 2 + 2C_1 + C_2 \end{aligned}$$

or, rearranging,

$$\begin{aligned} C_1 + C_2 &= -16 \\ 2C_1 + C_2 &= -8. \end{aligned}$$

Subtracting the first equation from the second gives $C_1 = 8$. Then $C_2 = -24$.

(G) So the solution of the initial-value problem is

$$y = 4t^2 + 12t + 14 + 2e^{-t} + 8e^{2t} - 24e^t$$