1 Link to Previous Diary

Click [here](#) for the older diary entries.

2 Assignment #6: First Problem

1 You will have been notified by email whether your solution to the first problem in Assignment #6 was correct. If you haven’t been so notified, the following applies to your second crack at this problem:

2 Precede your solution by stating what you are to prove.

3 Do not compact your writeup. I need to be able to read it and have room to write helpful (we hope) comments:

   (a) Write logical paragraphs with about 2 centimeters’ vertical whitespace separating them.

   (b) Leave margins of 3 centimeters.

Perhaps you could show a good friend your paper and ask them if reading it carefully is an attractive proposition.

4 I am moving toward forbidding you the use of the pronoun “it”. You have one more chance: make sure the referent for any “it” you use is clear. How do folks do that, do you suppose? Is the referent of the immediately preceding “that” clear?

5 Got it?

6 It is a linearly independent subset of it.

7 The adjectives “singular” and “nonsingular” apply only to square matrices. The statements “$A$ is nonsingular” or “$A$ is singular” or should not appear in this problem’s solution.

8 Do not attribute linear independence to a matrix. Do not attribute linear dependence to a matrix. Attribute these properties to sets of vectors. For example: “The set of rows of matrix $M$ forms a linearly independent set.” The abbreviations “LI” and “LD” are acceptable.
9 Similarly, “true” and “false” for us are attributes of statements. We cannot apply them
to things which are not statements. In our setting, a matrix cannot be “true.” A
solution of a system of equations cannot be “false.”

10 Equations and systems of equations have solutions. Matrices and vectors are not entities
which have solutions.

11 Referring to a theorem only by number is not acceptable. You can write “by Theorem
324” and then quote it briefly either parenthetically or in a footnote.

12 The theorem which several of you cited acceptably as the “more-vectors-than-entries
theorem” does not apply to this situation so far as I can tell. Matrix $A$ has only
$(n - 1)$ columns. These columns are vectors in $\mathbb{R}^n$, so we have fewer vectors than
entries. The column set of matrix $B$ consists of just $n$ vectors in $\mathbb{R}^n$.

13 Hint: how does $A\vec{b}$ relate to the columns of $A$? Can you write down the in-context
dependence equation and provide it a non-trivial solution? Like, actual values for the
unknown scalars in the dependence equation?
3 2/9/07 - Friday - Day Fifteen

1. Click [here](#) for info on elementary matrices.

2. Corresponding to each ERO is an elementary matrix. And vice versa.

3. Elementary matrices are invertible.

4. We made progress on adding another item to the list of things that are logically equivalent to nonsingularity: if a matrix is row-equivalent to the identity, then it can be written as a finite product of elementary matrices.

5. If a matrix is a product of elementary matrices...
2/12/07 - Monday - Day Sixteen

Kerr ain’t so sure as to the day we covered these topics:

1. Connecting geometric vectors to algebraic vectors.

2. Multiplying a vector by a scalar, an arrow picture. This gives the parallelism criterion for algebraic vectors.

3. Adding vectors together to get the basic parallelogram.
5 2/13/07 - Tuesday - Day Seventeen

Kerr ain’t so sure as to the day we covered these topics:

1. The dot product.
2. Connecting the dot product to the length of a vector and to matrix multiplication.
3. The polar decomposition of a vector:

\[ \bar{a} = ||\bar{a}|| \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix}, \]

where \( \alpha \) is the angle between the positive \( x \)-axis and \( \bar{a} \).

4. \( \bar{a} \cdot \bar{b} = \bar{b} \cdot ||\bar{a}|| ||\bar{b}|| \cos(\alpha - \beta) = ||\bar{a}|| ||\bar{b}|| \cos(\beta - \alpha) \), where \( \alpha - \beta \) or \( \beta - \alpha \) is the angle between \( \bar{a} \) and \( \bar{b} \).

5. The absolute value of the determinant of the matrix with \( \bar{a} \) and \( \bar{b} \) as columns gives the area of the basic parallelogram for the vectors \( \bar{a} \) and \( \bar{b} \).

Can you see how this computation tells us to arrange \( \bar{a} \) and \( \bar{b} \) as columns so as to get a positive determinant?
6 2/14/07 - Wednesday - Day Eighteen

1 A summary of lore so far of the Basic Parallelogram for $\mathbb{R}^2$ vectors $\vec{a}$ and $\vec{b}$.
   (a) Diagonals $\vec{a} + \vec{b}$ and $\vec{a} \pm \vec{b}$.
   (b) The angle between $\vec{a}$ and $\vec{b}$ and the dot product.
   (c) The determinant of the $2 \times 2$ matrix whose columns are $\vec{a}$ and $\vec{b}$ relates to the area of the Basic Parallelogram.
   (d) How to spot the sign of the above determinant from the relative positions of $\vec{a}$ and $\vec{b}$.

2 The algebraic
   (a) parallelism criterion for vectors (via multiplication by a scalar);
   (b) perpendicularity criterion for vectors (via the dot product).

3 Line Equation Type I: based on the perpendicularity criterion (Vector-Normal Form):
   (i) $3x + 4y = 24$ rewrites as $3(x + 8) + 4(y - 12) = 0$. Corrected
   (ii) That is,
        \[
        \begin{bmatrix}
        3 \\
        4
        \end{bmatrix}
        \cdot
        \begin{bmatrix}
        x + 8 \\
        y - 12
        \end{bmatrix}
        = 0,
        \]
        and that is the vector from $(-8, 12)$ to $(x, y)$ must be perpendicular to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in order that $(x, y)$ lie on the line in question. Corrected
   (iii) So, one can immediately grok $3x + 4y = 24$ as a line with normal vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
   (iv) The line parallel to $3x + 4y = 24$ and passing through the point $(-2, 5)$ has an equation of the form
        \[
        \begin{bmatrix}
        3 \\
        4
        \end{bmatrix}
        \cdot
        \begin{bmatrix}
        x \\
        y
        \end{bmatrix}
        = \begin{bmatrix}
        3 \\
        4
        \end{bmatrix}
        \cdot
        \begin{bmatrix}
        -2 \\
        5
        \end{bmatrix}
        \]
        or $3x + 4y = 14$.
   (v) The line perpendicular to $3x + 4y = 24$ and passing through the point $(-2, 5)$ has an equation of the form
        \[
        \begin{bmatrix}
        4 \\
        -3
        \end{bmatrix}
        \cdot
        \begin{bmatrix}
        x \\
        y
        \end{bmatrix}
        = \begin{bmatrix}
        4 \\
        -3
        \end{bmatrix}
        \cdot
        \begin{bmatrix}
        -2 \\
        5
        \end{bmatrix}
        \]
        or $4x - 3y = -23$. 

4 Line Equation Type II: based on the parallelism criterion (Vector-Parallel Form).

(i) The line \(3x + 4y = 24\) passes through the points \((8, 0)\) and \((0, 6)\) and hence is parallel to \([-8 \ 6]\).

(ii) If \((x, y)\) lies on this line, then its position vector, \([x \ y]\), is given by

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8 \\ 6 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix} + t \begin{bmatrix} -8 \\ 6 \end{bmatrix},
\]

which says that, to get to \((x, y)\) from the origin, you go first to \((8, 0)\), and then go along a line parallel to \([-8 \ 6]\) until you arrive at \((x, y)\).

5 Right-handed coordinates (front-left corner) for \(\mathbb{R}^3\)
7  2/16/07 - Friday - Day Nineteen

1  Extend the dot product to $\mathbb{R}^3$. It still relates to vector length, and, by means of the law of cosines, relates to angle.

2  $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos(\theta)$, where $\theta$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

3  In $\mathbb{R}^2$, we had two ways of looking at lines: vector-parallel and vector-normal. The vector-parallel line-equation idea scales up to $\mathbb{R}^3$ just fine, but the vector-normal idea gets us planes in $\mathbb{R}^3$. 
8 2/20/07 - Tuesday - Day Twenty

1 The orthogonal projection, $\text{proj}_\bar{v} (\bar{u})$ of $\bar{u}$ onto $\bar{v}$ has a short-cut formula:

$$\text{proj}_\bar{v} (\bar{u}) = \frac{\bar{u}^T \bar{v}}{\bar{v}^T \bar{v}} \bar{v}$$

Example: if $\bar{u} = \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, then

$$\text{proj}_\bar{v} (\bar{u}) = -\frac{7}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

2 The orthogonal complement of $\bar{u}$ relative to $\bar{v}$ is given by

$$\text{orth}_\bar{v} (\bar{u}) = \bar{u} - \text{proj}_\bar{v} (\bar{u})$$

and, in the example above, $\text{orth}_\bar{v} (\bar{u}) = \frac{1}{9} \begin{bmatrix} 41 \\ 52 \\ 67 \end{bmatrix}$

3 The distance from the line through $(2, 3, 9)$ and $(4, 5, 17)$ to the origin?

The line has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{u} + t\bar{m},$$

where $\bar{u} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$ and $\bar{m} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.

We drew pictures showing the line, the origin, $\bar{u}$, $\bar{m}$, $\text{proj}_{\bar{m}} (\bar{u}) = \frac{41}{18} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and

$$\text{orth}_{\bar{m}} (\bar{u}) = \frac{1}{18} \begin{bmatrix} -5 \\ 13 \\ -2 \end{bmatrix}.$$}

Our pictures showed that this last vector can be thought of as reaching from the origin out to the line, and perpendicular to the line. Its length, $\frac{\sqrt{22}}{6}$, is the sought distance. $\text{orth}_{\bar{m}} (\bar{u})$ is the position vector of the point of the line that is closest to the origin.
Let \( \mathbf{N} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} \). The plane perpendicular to \( \mathbf{N} \) through the point \( (5, 9, -7) \) has an equation given by

\[
\mathbf{N} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{N} \cdot \begin{bmatrix} 5 \\ 9 \\ -7 \end{bmatrix}.
\]

This makes the plane out to be the set of all points whose position vectors have projections onto \( \mathbf{N} \) all equal to the projection of the position vector of \( (5, 9, -7) \) onto \( \mathbf{N} \).

5. Lookahead: what is the distance from the point \( (2, 3, 1) \) to the above plane?

Kerr believes this distance is \( \frac{46}{9} \).
9 2/21/07 - Wednesday - Day Twenty-One

1. We used the projection idea to compute the distance from \((2, 3, 1)\) to the plane \(4x + y + 8z = -27\).

One can find a point on the plane, \((0, -27, 0)\) say, and let \(\vec{u}\) be the vector from \((2, 3, 1)\) to \((0, -27, 0)\). Then

\[
\vec{u} = \begin{bmatrix} -2 \\ -30 \\ -1 \end{bmatrix}
\]

and the sought distance is \(||\text{proj}_N(\vec{u})||\).

Now

\[
\text{proj}_N(\vec{u}) = \frac{\vec{N} \cdot \vec{u}}{\vec{N} \cdot \vec{N}} \vec{N} = \frac{-46}{81} \vec{N},
\]

and so the distance we want is

\[
||\text{proj}_N(\vec{u})|| = \frac{46}{81} ||\vec{N}|| = \frac{46}{9}
\]

2. We convinced ourselves that the set of all linear combinations of the vectors

\[
\vec{a} = \begin{bmatrix} 7 \\ 4 \\ 7 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix}
\]

must be a plane through the origin. This plane must have an equation of the form

\[
Ax + By + Cz = 0.
\]

Imposing the points \((7, 4, 7)\) and \((2, 0, -9)\) led to the system

\[
\begin{align*}
2A - 9C &= 0 \\
7A + 4B + 7C &= 0
\end{align*}
\]

whose augmented matrix is row equivalent to the RREF

\[
\begin{array}{ccc|c}
1 & 0 & -9/2 & 0 \\
0 & 1 & 77/8 & 0 \\
\end{array}
\]

from which we extracted the non-trivial solution \(A = 36, B = -77,\) and \(C = 8\). This gives a vector-normal equation for the plane:

\[
36x - 77y + 8z = 0
\]
3 The cross product of two vectors:

\[ \vec{x} \times \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix} . \]

so that

\[ \begin{bmatrix} 2 \\ 0 \\ -9 \end{bmatrix} \times \begin{bmatrix} 7 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 36 \\ -77 \\ 8 \end{bmatrix} . \]

4 \( \vec{x} \times \vec{y} = - (\vec{y} \times \vec{x}) \)

5 \( \vec{x} \perp \vec{x} \times \vec{y} \)

6 A sort of hokey argument favored the right-hand rule for determining the direction of \( \vec{x} \times \vec{y} \)

7 Lookahead:

(i) Solve for \( \vec{x} : \vec{a} \times \vec{x} = \vec{b} \)

(ii) Do the lines in problem 7, page 157, cross?
10 2/23/07 - Friday - Day Twenty-Two Test #1

Click [here](#) to see the test.

11 2/26/07 - Monday - Day Twenty-Three

1 We checked to see whether the lines

\[ \vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \]

cross. Owing to the inconsistency of the system

\[ \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} + u \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \]

we concluded they did not cross. Had this system _had_ a solution \( t = t_0 \) and \( u = u_0 \), then we could have found the crossing point \((p, q, r)\) by

\[ \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + t_0 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix} + u_0 \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \]

2 A more geometric approach to whether these lines cross goes like this:

If they _do_ cross, then they must form a plane. A normal vector for this plane is given by the cross product of the “slope vectors” of the two lines

\[ \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \]

So we can use \( \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \) as this plane’s normal vector.

If these lines _did_ cross, then the points \( A = (1, 5, 4) \) and \( B = (7, 2, 3) \) would lie in the plane they form. This means the vector \( \overrightarrow{AB} \), given by \( \overrightarrow{AB} = \begin{bmatrix} 6 \\ -3 \\ -1 \end{bmatrix} \), would be parallel to the plane, and so orthogonal to its normal vector. But

\[ \overrightarrow{AB} \cdot \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 12 \neq 0, \]

contradicting the existence of a plane formed by these two lines. So they must not cross.
3 We showed that the line \( \vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} \) meets the plane \( 36x - 77y + 8z = 0 \)

at the point with position vector \( \frac{1}{287} \begin{bmatrix} 921 \\ 484 \\ 514 \end{bmatrix} \)

4 The solution set of

\[
\begin{align*}
    x + 2y + 3z &= -4 \\
    4x + 5y + 6z &= -7 \\
    7x + 8y + 9z &= -10
\end{align*}
\]

is the straight line

\( \vec{x} = \begin{bmatrix} 8 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \),

that is, one of the three planes of the system contains the line of intersection of the other two planes.

5 We began “appreciating” the page-168 properties of \( \mathbb{R}^n \) and making as though they could be taken as the definition of an **Abstract Vector Space** (click [here](#)).
12 2/27/07 - Tuesday - Day Twenty-Four

1 We finished appreciating the vector-space axioms on page 168.

2 Some examples of vector spaces:

   (a) The set of all $2 \times 2$ matrices with real-number entries. We talked about how linear combinations and the dependence equation would look in this situation.

   (b) The space of real-valued functions continuous on $[0, 1]$.

   (c) The set $P_4$ of polynomials of degree at most four.

3 A subspace $W$ of a vector space $V$ is a subset of $V$ which is a vector space using the same operations as $V$.

4 Lookahead: as it stands, to verify that some $W$ is a subspace, we have to check all of the page-168 axioms. We were embroiled in trying to whittle down the list of things to check, when we noticed that we need to infer some theorems from the page-168 axioms:

   (a) For all $\vec{v}$ in $V$, $0\vec{v} = \vec{0}$.

   (b) The additive identity is unique in a vector space. We proved this one in class.

   (c) The additive inverse is unique in a vector space.
13 2/28/07 - Wednesday - Day Twenty-Five

We finished proving the theorem:

Let $V$ be an abstract vector space, and let $W$ be a subset of $V$. If $W$ does both of

(i) $W \neq \emptyset$,
(ii) $W$ is closed under linear-combination formation,

then $W$ is a subspace of $V$. 
14 3/2/07 - Friday - Day Twenty-Six

Today we group worked problem 31, page 175, using the black page-171 steps as a guide to proving that some set is a subspace.
15 3/5/07 - Monday - Day Twenty-Seven

1 We talked about how, in problem 4a of Test #1, solving the dependence equation

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} +
\begin{bmatrix}
1 & 3 \\
0 & 2 \\
\end{bmatrix} +
\begin{bmatrix}
1 & 9 \\
0 & 4 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

can be changed into solving the system

\[
\begin{align*}
c_1 + c_2 + c_3 &= 0 \\
3c_2 + 9c_3 &= 0 \\
c_1 + 2c_2 + 4c_3 &= 0
\end{align*}
\]

which has coefficient matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 3 & 9 \\
1 & 2 & 4 \\
\end{bmatrix}
\]

2 We studied the nullspace, \( \mathcal{N}(A) \) of an \( m \times n \) matrix \( A \). We proved that the nullspace of a matrix is always a subspace of \( \mathbb{R}^n \).

3 We studied the span, \( \text{Sp}(S) \) of a subset \( S \) of a vector space. The span of a set is always a subspace.

4 We showed that the nullspace of the problem-4a matrix is the span of the set consisting of the vector \( \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \). This meant that there is a non-trivial solution to the dependence equation above, so that the set of three \( 2 \times 2 \) matrices is linearly dependent.
16 3/6/07 - Tuesday - Day Twenty-Eight

1 We showed that the Range, $\mathcal{R}(A)$, of the $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$. We showed this two ways:

(a) Directly from the definition of $\mathcal{R}(A)$, via the page-171 black box.

(b) By observing that $\mathcal{R}(A)$ can be expressed as the span of some set, the column set of $A$.

2 We showed that \[
\begin{bmatrix}
10 \\
8 \\
3
\end{bmatrix}
\]
is not in the span of the set
\[
\left\{ \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}
\]

3 At the end of the hour we were looking at the Row Space of $n \times n$ matrix $A$. We were trying to pay dues as to the proposition that EROs done to a matrix do not change its row space.

4 Lookahead: How many subspaces does $\mathbb{R}^2$ have? How many of them are linearly independent sets?
17  3/7/07 - Wednesday - Day Twenty-Nine

Mon Apr 2 16:42:50 MDT 2007

1  An elementary-matrix proof of Theorem 6, section 3.3: row-equivalent matrices have identical row space.

2  Like unto example 5, page 184: given

\[ \begin{bmatrix}
-2 \\
3 \\
-2 \\
-1
\end{bmatrix}, \begin{bmatrix}
-1 \\
2 \\
-1 \\
3
\end{bmatrix}, \begin{bmatrix}
3 \\
-3 \\
-2 \\
-4
\end{bmatrix},
\]

and

\[ \begin{bmatrix}
-3 \\
5 \\
-2 \\
0
\end{bmatrix}, \]

let \( W = \text{Sp}(\{ \vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4 \}) \).

We defined a matrix \( A \) with columns \( \vec{u}_1, \vec{u}_2, \vec{u}_3, \) and \( \vec{u}_4 \). Then we row reduced \( A^T \) thereby arriving at a basis for the row space of \( A^T \). Transposing the non-zero rows of the reduced echelon form of \( A^T \) yields a basis \( \vec{v}_1, vev_2, \) and for \( \text{Sp}(\{ \vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4 \}) \)

\[ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 5/4 \end{bmatrix}. \]

3  We found a spanning set for the range, \( \mathcal{R}(A) \), of matrix \( A_i \) given by

\[ A = \begin{bmatrix}
1 & 1 & 1 \\
0 & 3 & 9 \\
1 & 2 & 4
\end{bmatrix}, \]

through the expedient of augmenting \( A \) with a \( 3 \times 3 \) identity matrix (for the entries \( b_1, b_2, \) and \( b_3 \) of the vector \( \vec{b} \) such that \( A\vec{x} = \vec{b} \)). This \( 3 \times 6 \) matrix row reduced to

\[ \begin{bmatrix}
1 & 0 & -2 & 1 & -1/3 & 0 \\
0 & 1 & 3 & 0 & 1/3 & 0 \\
0 & 0 & 0 & -1 & -1/3 & 1
\end{bmatrix}, \]

from which we infer that, for \( \vec{b} \) to belong to \( \mathcal{R}(A) \), we must have \(-b_1 - \frac{1}{3}b_2 + b_3 = 0\). That is,

\[ \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = u \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \\ 1/3 \end{bmatrix}. \]
So the range of $A$ is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1/3 \end{bmatrix}$$
3/9/07 - Friday - Day Thirty

To Appear!
19 3/12/07 - Monday - Day Thirty-One

1 The Coordinate Theorem under Uniqueness of Representation on page 195.

2 The theorem on forming a basis of $\mathcal{R}(A)$ using just columns of $A$. This theorem would help one reduce a spanning set to a basis.

**Lemma A** If $B$ is an $n \times n$ nonsingular matrix, and if $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly independent, then $\{B\vec{v}_1, \ldots, B\vec{v}_p\}$ is also linearly independent.

**Proof:** ELTS

The main theorem:

Let non-zero $m \times n$ matrix $A$ be row equivalent to reduced-echelon matrix $R$. The columns of $A$ corresponding to leading-1 columns of $R$ furnish a basis for the column space, $\mathcal{R}(A)$, of $A$.

**Proof:** If $R$ has $r$ non-zero rows, then $R$ has $r$ leading-1 columns, $\vec{e}_1, \vec{e}_2, \ldots \vec{e}_r$, which are the first $r$ columns of the $m \times m$ identity matrix.

For $1 \leq i \leq r$, let $R_{k_i} = \vec{e}_i$, the column of $R$ that harbors $\vec{e}_i$.

The other columns of $R$ are linear combinations of these $\vec{e}_i$. For each $R_j$, there are scalars $a_1, \ldots, a_q$, the first several entries of $R_j$, such that

$$R_j = \sum_{i=1}^{q} a_i R_{k_i}.$$  

Thus the set $E$ of $R_{k_i}$ spans the column space of $R$. Since $E$ is linearly independent, it forms a basis for the column space of $R$.

Now we use this set $E$ to cook up a basis for $\mathcal{R}(A)$. Let $M$ be the nonsingular product of elementary matrices which yields $R = MA$. $M$ comes from the EROs which transform $A$ into $R$.

Let $X \in \mathcal{R}(A)$. Then $MX$ is in the column space of $R$, and hence, since $E$ is a basis,

$$MX = \sum_{i=1}^{r} x_i R_{k_i},$$

for scalars $x_i$. But this means that

$$X = M^{-1} \sum_{i=1}^{r} x_i R_{k_i} = \sum_{i=1}^{r} x_i M^{-1} R_{k_i} = \sum_{i=1}^{r} x_i A_{k_i},$$

that is, we can write $X$ as a linear combination of the columns $A_{k_i}$ of $A$. 
Since $X$ is an arbitrary element of $\mathcal{R}(A)$, this means that the set of $A_{k_i}$ must span $\mathcal{R}(A)$.

The Lemma above assures us that the set of $A_{k_i}$ is also linearly independent, so we know this set constitutes a basis for $\mathcal{R}(A)$. 
20  3/13/07 - Tuesday - Day Thirty-Two

1  To Appear!
3/14/07 - Wednesday - Day Thirty-Three

To Appear!
22  3/16/07 - Friday - Day Thirty-Four

1  To Appear!
23  3/19/07 - Monday - Day Thirty-Five

1  A version of Definition 8, page 226:
   Let $V$ and $W$ be subspaces of unspecified vectors spaces. Then $T$ is a **Linear Transformation** from $V$ to $W$ if
   (i) $T$ is a function from $V$ to $W$.
   (ii) The domain of $T$ is all of $V$.
   (iii) For all $\vec{x}$ and $\vec{y}$ in $V$ and for all scalars $\alpha$ and $\beta$ we have

   $$T(\alpha \vec{x} + \beta \vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$$

2  (Cf Examples 1 and 2, page 227-8)

   (A) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 3x_2 \end{bmatrix}$

   One can check the definition to see that this is a linear transformation. Alternatively, one can observe that

   $$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   Since matrix mutiplication does $A(\alpha \vec{x} + \beta \vec{y}) = \alpha A \vec{x} + \beta A \vec{y}$, we have $T$ is linear because it can be realized via matrix multiplication.

   (B) Lemma: If $T$ is linear, then $T(\vec{\theta}) = \vec{\theta}$.

   This lemma showed that $T$, given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2 \\ x_1 + x_2 + 3 \\ 3x_2 \end{bmatrix}$, cannot be linear.

   (C) Lemma, if $T$ is linear, then, for all real $\alpha$ and all $\vec{x}$ in the domain of $T$, we have $T(\alpha \vec{x}) = \alpha T(\vec{x})$

   This lemma shows that $T$, given by

   $$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_2^3 \end{bmatrix}$$

   cannot be linear.

3  A mathematical induction proof that if $T$ is linear,

   $$T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T(\vec{v}_i)$$

   for all positive integers $n$. 

4. The values of a linear transformation are completely determined by its values on a basis for its domain space.
24 3/20/07 - Tuesday - Day 36

1 Proof of the “subset-of” theorem

If $U$ and $V$ are subspaces of the same vector space,

$$\text{Sp}(U \cup V) = U + V$$

2 Dot product extends to $\mathbb{R}^Q$.

3 Definition of orthogonal set of vectors in $\mathbb{R}^Q$.

4 Definition of Kronecker’s Delta

5 Definition of orthonormal set of vectors in $\mathbb{R}^Q$.

6 An orthogonal set is linearly independent.

7 Coordinates relative to a basis and coordinates relative to an orthonormal basis.

8 Extend the idea of projection to $\mathbb{R}^Q$:

$$\text{proj}_{\vec{x}}(\vec{y}) = \left(\frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}\right) \vec{x}$$

9 $$(\vec{y} - \text{proj}_{\vec{x}}(\vec{y})) \cdot \vec{x} = \vec{0}$$
25 3/21/07 - Wednesday - Day Thirty-Seven

1 If \( \{\vec{q}_1, \ldots, \vec{q}_p\} \) is an orthogonal set in \( \mathbb{R}^n \), then, for any vector \( \vec{v} \) in \( \mathbb{R}^n \) we have, for all \( \ell \),

\[
\vec{q}_\ell \perp \left( \vec{v} - \sum_{i=1}^{p} \text{proj}_{\vec{q}_i} (\vec{v}) \right)
\]

2 See the 4/2/07 entries.

26 3/23/07 - Friday - Day Thirty-Eight

1 Test #2
27 4/2/07 - Monday - Day Thirty-Nine

1 Replay, reconsider from Wednesday, 3/21/07: Let \( \{ \vec{v}_1, \ldots, \vec{v}_p \} \) be a basis for subspace \( W \) in \( \mathbb{R}^Q \). Let
\[
\vec{u}_1 = \vec{v}_1
\]
and, for \( 2 \leq k \leq p \), let
\[
\vec{u}_k = \vec{v}_k - \sum_{i=1}^{k-1} \text{proj}_{\vec{u}_i}(\vec{v}_k)
\]
We proved that the \( \vec{u}_k \) vectors generated by this process form an orthogonal basis for subspace \( W \). The heart of this proof was a mathematical induction argument that, for \( 1 \leq n \leq p \),
\[
\text{Sp}(\{ \vec{u}_1, \ldots, \vec{u}_n \}) = \text{Sp}(\{ \vec{v}_1, \ldots, \vec{v}_n \})
\]

2 An orthogonal basis for the space spanned by \( \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \), and
\[
\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]
is given by \( \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \), \( \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \), and \( \vec{u}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ -4 \\ 1 \end{bmatrix} \).

3 An orthonormal basis for this same subspace is given by \( \vec{n}_1 = \frac{1}{2} \vec{u}_1 \), \( \vec{n}_2 = \frac{1}{\sqrt{28}} \vec{u}_2 \),
and \( \vec{n}_3 = \frac{1}{\sqrt{70}} \vec{u}_3 \),

4 Why folks like orthonormal bases:

(a) Ease of computing coordinates relative to an orthonormal basis.

(b) If \( \{ \vec{n}_1, \ldots, \vec{n}_p \} \) is an orthonormal basis and if \( \vec{a} = \sum_{i=1}^{p} a_i \vec{n}_i \) and \( \vec{b} = \sum_{i=1}^{p} b_i \vec{n}_i \),
then
\[
\vec{a} \cdot \vec{b} = \sum_{i=1}^{p} a_i b_i
\]

(c) \( ||\vec{a}||^2 = \sum_{i=1}^{p} a_i^2 \)
(d) Lookahead: a MATH-275 example: the speed of a bug moving according to the position function

\[ \vec{v}(t) = \begin{bmatrix} e^{2t} \cos(3t) \\ e^{2t} \sin(3t) \\ 5t \end{bmatrix} \]

28 Link to Previous Diary

Click [here](#) for the older diary entries.