Mon Feb 12 13:58:30 MST 2007 /m301.sp07/handouts301/Diary301/DiaryStar211

1 Current Diary

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2 1/16/07 - Tuesday - Day One

(1) We used **Elementary Operations** (aka EROs – bottom of page 8) and the **augmented matrix** to solve

$$\begin{aligned} \boldsymbol{x_1+3x_2+x_3} &= \mathbf{1} \\ \boldsymbol{2x_1+7x_2+x_3-x_4} &= \mathbf{-1} \\ \boldsymbol{3x_1-2x_2+4x_4} &= \mathbf{8} \\ \boldsymbol{-x_1+x_2-3x_3-x_4} &= \mathbf{-6} \end{aligned}$$

We got one solution:

$$\boldsymbol{x_1 = 1/2 \qquad x_2 = -1/4 \qquad x_3 = 5/4 \qquad x_4 = 3/2.}$$

The method we used is known as the **Gauss-Jordan Elimination Method,** a major MATH-301 workhorse.

(2) We also spent time on the simpler system

$$\begin{aligned} \boldsymbol{x_1+2x_2+3x_3} &= \mathbf{3} \\ \boldsymbol{4x_1+5x_2+6x_3} &= \mathbf{3} \\ \boldsymbol{7x_1+8x_2+9x_3} &= \mathbf{3} \end{aligned}$$

which has a final augmented matrix

$$\left[\begin{array}{ccr|r} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

from which we extracted a formula for the *infinitely many* solutions:

$$\begin{aligned} \boldsymbol{x_1} &= \boldsymbol{-3+t} \\ \boldsymbol{x_2} &= \boldsymbol{3-2t} \\ \boldsymbol{x_3} &= \boldsymbol{t} \end{aligned}$$

This happened because we could not eliminate dependencies among the variables. We could not isolate each variable in its own equation as we were able to do in the first four-equations-in-four-unknowns example.

(3) In the last two or three minutes of class we looked at the system

$$\begin{aligned} \boldsymbol{x_1 + 2x_2 + 3x_3} &= \mathbf{3} \\ \boldsymbol{4x_1 + 5x_2 + 6x_3} &= \mathbf{3} \\ \boldsymbol{7x_1 + 8x_2 + 9x_3} &= \mathbf{5280.} \end{aligned}$$

We EROed this until we got

$$\left[\begin{array}{rrr|r} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{5277} \end{array}\right],$$

which indicates that the given system has no solutions whatsoever.

(4) Looking ahead: we will need to check our answer to the $\mathbf{4 \times 4}$ system (pages 51-54) or is it $\mathbf{4 \times 5}$?). And we have to justify use of the EROs (1.1: problem 41 describes this).

3 1/17/07 - Wednesday - Day Two

(1) We showed a way to check the solution we found in our very first example on Tuesday.

(2) We had to justify use of the EROs. We did part of Theorem 1.1 which says that the application of an ERO to a system produces a system with *exactly the same solution set.* We went through the $E_i \leftarrow E_i + k\,E_j$ part of the proof ($i \neq j$).

(3) We must worry that the EROs might alter the solution set, as happens when we use the transformation "square both sides" on an equation like

$$x + 2 = -\sqrt{x+4}.$$

(4) The proof used the fact that every ERO is reversible: given a particular ERO, one can find an ERO (¿always different from the first ERO?) which undoes the action of the first ERO.

(5) On the notation front, equation E_i is a *linear* equation:

$$\sum_{r=1}^{n} a_{ir}x_r = b_i$$

(6) The clock ran out on matrix indices (Cf problems 1.1: 7-10).

(7) Looking ahead: **echelon matrix** and **echelon form** of a matrix. And the real deal: the **reduced echelon form** of a matrix. This is known also as the **row reduced echelon form** or the **reduced row echelon form.** This is why your calculator has an `rref` command and in scilab we get

```
-->A =  [1 2 3 3; 4 5 6 3; 7 8 9 3]
A  =
    1.    2.    3.    3.
    4.    5.    6.    3.
    7.    8.    9.    3.

-->rref(A)
 ans  =

    1.    0.  - 1.  - 3.
    0.    1.    2.    3.
    0.    0.    0.    0.
```

4 1/19/07 - Friday - Day Three

(1) Page 10: two matrices are **Row Equivalent**

(2) Without supporting computations it was stated that an **Echelon Form** of the augmented matrix of

$$\begin{aligned} 2x_1 - 6x_2 + 3x_3 - 2x_4 &= -1 \\ -x_1 + 3x_2 - 2x_3 &= 4 \\ 3x_1 - 9x_2 + 4x_3 - 4x_4 &= 2 \end{aligned}$$

turns out to be

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & 0 & -4 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Another ERO step brings us to the **Reduced Echelon Form** (aka RREF),

$$\left[\begin{array}{rrrr|r} 1 & -3 & 0 & -4 & 10 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which is a real workhorse for subsequent developments. Check out the definition and be able to compute it by hand. And note that `scilab` has an “rref” command. So does Maple.

(3) The above RREF shows that x_2 and x_4 are so-called **free variables** because they have messy columns. x_1 and x_3 are sometimes called **basic** variables. They are the variables corresponding to left-most non-zero entries in the RREF matrix. They have *nice* columns.

(4) The RREF greatly simplifies writing the formula for the many solutions that our example has. RREF leaves us with the two equations

$$\begin{aligned} x_1 - 3x_2 - 4x_4 &= 10 \\ x_3 + 2x_4 = -7, \end{aligned}$$

which we crowded into this form:

$$\begin{aligned} x_1 &= 10 + 3u + 4v \\ x_2 &= u \\ x_3 &= -7 - 2v \\ x_4 &= v \end{aligned}$$

which we massaged into a column matrix form:

$$\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix} = \begin{bmatrix} \mathbf{10} \\ \mathbf{0} \\ \mathbf{-7} \\ \mathbf{0} \end{bmatrix} + \boldsymbol{u} \begin{bmatrix} \mathbf{3} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \boldsymbol{v} \begin{bmatrix} \mathbf{4} \\ \mathbf{0} \\ \mathbf{-2} \\ \mathbf{1} \end{bmatrix}$$

5 1/22/07 - Monday - Day Four

(1) We computed the crossing point of the lines

$$\begin{aligned} 2x + 5y &= 3 \\ 4x + 3y &= -8 \\ 2x + 2y &= -3 \end{aligned}$$

by racking their equations up in an augmented matrix for which we found the RREF:

$$\left[\begin{array}{cc|r} 1 & 0 & -7/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

which gave us the crossing point at $(-7/2, 2)$.

(2) Some folks might say that the previous system has infinitely many solutions, owing to the row of zeros. How would you talk them out of that view?

(3) We took three points $(2, 10)$, $(1, 3)$, $(-6, 4)$ and found an equation of form

$$x^2 + y^2 + Ax + By + C = 0$$

for the circle through those points:

$$(x + 2)^2 + (y - 7)^2 = 25$$

with center at $(-2, 7)$ and radius 5.

(4) We also did a partial-fractions decomposition. We guessed that

$$\mathcal{F} = \frac{5x}{x^3 + 2x^2 - x - 2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Comparing the identically-equal numerators on both sides,

$$\begin{aligned} 5x + 7 &= = A(x-1)(x+1) + B(x+2)(x+1) + C(x+2)(x-1) \\ &= x^2(A + B + C) + x(3B + C) + (-A + B - 2C), \end{aligned}$$

we arrive at a system of equations:

$$\begin{aligned} A + B + C &= 0 \\ 3B + C &= 5 \\ -A + B - 2C &= = 7 \end{aligned}$$

for which $A = 2$, $B = -1$, and $C = -1$. Thus

$$\mathcal{F} = \frac{2}{x+2} - \frac{1}{x-1} - \frac{1}{x+1}.$$

(5) Look ahead:

  (i) On Tuesday we'll begin by seeking the line of form $\boldsymbol{Ax + By + C = 0}$ which passes through the three points $\mathbf{(-286, 10),}$ $\mathbf{(434, -15),}$ and $\mathbf{(146, -5).}$ This relates to example 8 in 1.3.

  (ii) What happens if we use the points $\mathbf{(2, 10),}$ $\mathbf{(1, 3),}$ and $\mathbf{(-6, 4)}$ from the circle problem above?

  (iii) We'll wax theoretical from the material in 1.3, pages 28-32.

6 1/23/07 - Tuesday - Day Five

(1) We looked at the line of form $\boldsymbol{Ax + By + C = 0}$ through the points $\boldsymbol{(-286, 10)}$, $\boldsymbol{(434, -15)}$, and $\boldsymbol{(146, -5)}$.

Descartes says that these points give rise to the **homogeneous** system

$$\begin{aligned} -286A + 10B + C &= 0 \\ 434A - 15B + C &= 0 \\ 146A - 5B + C = 0. \end{aligned}$$

We racked up an augmented matrix with the variables in $\boldsymbol{CBA}$ order and EROed into **reduced echelon form:**

$$\left[\begin{array}{ccr|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -144/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this we were able to see that the homogeneous system in question has **non-trivial** solutions. Our variable-order choice makes $\boldsymbol{A}$ the independent (free) variable:

$$\begin{aligned} A &= t \\ B &= 114/5t \\ C &= -2t \end{aligned}$$

or

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = t \begin{bmatrix} 1 \\ 144/5 \\ -2 \end{bmatrix} = u \begin{bmatrix} 5 \\ 144 \\ -10 \end{bmatrix}.$$

We wrote down a line equation: $\boldsymbol{5x + 144y - 10 = 0}$, one of the infinitely many equations of this form for the line through these three points.

(2) We know that the points $\boldsymbol{(2, 10)}$, $\boldsymbol{(1, 3)}$, and $\boldsymbol{(-6, 4)}$ are non-collinear. Had we used them above, the corresponding homogeneous system would have been consistent. So, what line equation would it give for these points? That is, how would this computation tip us off to non-collinearity.

(3) What would happen had we used our Monday method to find an equation for the circle through $\boldsymbol{(-286, 10)}$, $\boldsymbol{(434, -15)}$, and $\boldsymbol{(146, -5)}$? How would the calculation have tipped us off? To what?

(4) We read through the section 1.3 remarks on page 29 ff.

(5) Look ahead: Example 7, page 33 and traffic flow and Kirchoff.

7 1/24/07 - Wednesday - Day Six

(1) We did an example like example 7, page 33. What must we have for $\boldsymbol{b_1}$, $\boldsymbol{b_2}$, and $\boldsymbol{b_3}$ in order that the following system be **consistent**?

$$\begin{aligned} \boldsymbol{x+2y+3z} &= \boldsymbol{b_1} \\ \boldsymbol{4x+5y+6z} &= \boldsymbol{b_2} \\ \boldsymbol{7x+8y+9z} &= \boldsymbol{b_3} \end{aligned}$$

We used variable order $\boldsymbol{xyzb_1b_2b_3}$ to rack this up in an AUGmented matrix:

$$\left[\begin{array}{ccc|ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right]$$

We EROed this only until the left half was RREFed:

$$\left[\begin{array}{ccr|rrc} \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{-5/3} & \mathbf{2/3} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{4/3} & \mathbf{-1/3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{array}\right].$$

The third row here shows us that consistency is guaranteed if

$$\boldsymbol{b_1 - 2b_2 + b_3 = 0.}$$

In that event, a solution formula is given by

$$\begin{aligned} \boldsymbol{x_1} &= \boldsymbol{t - 5/3b_1 + 2/3b_2} \\ \boldsymbol{x_2} &= \boldsymbol{-2t + 4/3b_1 - 1/3b_2} \\ \boldsymbol{x_3} &= \boldsymbol{t.} \end{aligned}$$

(2) We studied a traffic-flow example, here using Maple to study the many solutions. Click here.

8 1/24/07 - Friday - Day Seven

(1) We went through an establishment approved version of the proof in 1.1: 40.

(2) Henceforth in MATH 301 001 auxiliary variables will be used in reporting the solutions of a system with many solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

for 1.2: 50.

(3) Then we began our look at the stuff in sections 1.5 and 1.6 with the matrix product $A\vec{x} = \vec{y},$ where

(a) A is an $m \times n$ matrix,

(b) $\vec{x}$ is an $n \times 1$ column matrix, also known as an n-vector (and also $\vec{x} \in \mathbb{R}^n$

(c) $\vec{y} \in \mathbb{R}^m$

(d) The i^{th} entry of $\vec{y}$ is given by

$$y_i = \sum_{k=1}^{n} a_{ik} x_k$$

We discussed the usual arm-waving way of computing $A\vec{x}$. The summation-index formula needs to kept in mind for proofs and computer programs.

(4) We used the above product to define $AB,$ where B is not a column matrix. Let B have q columns $\left\{\vec{b}_1, \ldots, \vec{b}_q\right\}$. Then AB is the matrix

$$AB = \left[A\vec{b}_1 | A\vec{b}_2 | \cdots | A\vec{b}_{q-1} | A\vec{b}_q\right],$$

so that entry i of column j of AB is given by $\sum_{k=1}^{n} a_{ik} b_{kj},$ where a_{st} denotes the A-entry in row s and column t.

(5) So AB makes sense only if A is $m \times n$ and B is $n \times q$.

(6) We also looked at how one can take the rows $\{\boldsymbol{a_1}, \ldots, \boldsymbol{a_m}\}$ of $\boldsymbol{A}$ and think of $\boldsymbol{AB}$ by rows as

$$\boldsymbol{AB} = \begin{bmatrix} \boldsymbol{a_1 B} \\ \boldsymbol{a_2 B} \\ \cdots \\ \boldsymbol{a_{m-1} B} \\ \boldsymbol{a_m B} \end{bmatrix}$$

(7) Lookahead: interpretation of $\boldsymbol{A\vec{x}}$ as a **linear combination** of the columns $\{\boldsymbol{\vec{a}_1}, \ldots, \boldsymbol{\vec{a}_m}\}$ of $\boldsymbol{A}$. Also

- (a) Identity matrix $\boldsymbol{I_n}$
- (b) Noncommutativity of matrix multiplication
- (c) General index twiddling
- (d) Associativity
- (e) Divisors of Zero
- (f) Linear systems in form $\boldsymbol{A\vec{x}} = \boldsymbol{\vec{b}}$
- (g) Transpose
- (h)
- (i)

9 1/29/07 - Monday - Day Eight

(1) Click here for another POV on matrix multiplication.

(2) An example showing matrix multiplication is noncommutative.

(3) We reviewed the summation-and-index formulas for the entries of $\boldsymbol{A\vec{x}}$ and $\boldsymbol{AB}$.

(4) $\left(\boldsymbol{A^T}\right)_{\boldsymbol{ij}} =?$

(5) Index-wise proof that $\boldsymbol{A^T B^T = (BA)^T}$.

(6) Index-wise proof that matrix multiplication is associative.

(7) Definition of **linear combination.**

(8) $\boldsymbol{A\vec{x}}$ is a linear combination of the columns of $\boldsymbol{A}$.

(9) If

$$\begin{aligned} \boldsymbol{ax + by} &= \boldsymbol{E} \\ \boldsymbol{cx + dy} &= \boldsymbol{F} \end{aligned}$$

has just one solution, it is given by

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \frac{\boldsymbol{1}}{\boldsymbol{ad - bc}} \begin{bmatrix} \boldsymbol{d} & \boldsymbol{-b} \\ \boldsymbol{-c} & \boldsymbol{a} \end{bmatrix} \begin{bmatrix} \boldsymbol{E} \\ \boldsymbol{F} \end{bmatrix}$$

(10) Lookahead: linear independence and associated matters (see 1.7).

10 1/30/07 - Tuesday - Day Nine

(1) Using the matrix $\boldsymbol{A}$ from 1.3: 24, looked at expressing

$$\vec{b}_1 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$$

as **linear combinations** of the columns of $\boldsymbol{A}$.

Using the idea that

$$A\vec{x} = \sum_{i=1}^{3} x_i A_i,$$

where A_i is the i^{th} column of A, we saw that this problem was equivalent to the consistency of the systems

$$A\vec{x} = \vec{b}_1 \quad \text{and} \quad A\vec{x} = \vec{b}_2.$$

The second system came up inconsistent, so B_2 is not a linear combination of columns of A. But the first one is consistent – its RREF yielded a vector solution formula

$$\begin{aligned} x_1 &= 12 - 2t \\ x_2 &= -5 + t \\ x_3 &= t, \end{aligned}$$

from which we can get coefficients for the sought linear combination ($t = 6$):

$$b_1 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = (0) \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + (1) \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + (6) \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

(2) We saw that we could write the zero vector $\vec{\theta}$ as a non-trivial linear combination of the columns of A.

(3) This led to the definitions of **Linear Dependence** and **Linear Independence.** And the observation that the columns of A above constitute a linearly dependent set.

(4) We ended with a proof that a set of vectors in linearly dependent if and only if one of the vectors can be expressed as a linear combination of the other vectors in the set.

(5) Lookahead: In class, as an all-group project, we'll look into the page-78 problems:

(a) 1
(b) 2, 6
(c) $\{\boldsymbol{u_0}, \boldsymbol{u_1}\}$
(d) 11
(e) $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$
(f) 14

and $\mathbb{R}^{\boldsymbol{n}}$**.**

11 1/31/07 - Wednesday - Day Ten

(1) We spent time on the page-78 problems from the Monday lookahead.

  (a) **Theorem** $\{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}\}$ is linearly dependent $\Longleftrightarrow$ there is a scalar $\boldsymbol{k}$ such that $\vec{\boldsymbol{u}} = \boldsymbol{k}\vec{\boldsymbol{v}}$**.**

  (b) From page 78, $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is linearly dependent because

$$\boldsymbol{RREF}\left(\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{1} & \mathbf{4} \\ \mathbf{-1} & \mathbf{-3} & \mathbf{3} \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

  so that there must be not-all-zero $\boldsymbol{c_i}$ $(\boldsymbol{i} = \mathbf{1, 2, 3})$ such that

$$\boldsymbol{c_1}\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix} + \boldsymbol{c_2}\begin{bmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{-3} \end{bmatrix} + \boldsymbol{c_3}\begin{bmatrix} \mathbf{-1} \\ \mathbf{4} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \vec{\boldsymbol{\theta}}$$

  (c) In looking at these problems we bumbled into the famous **More-Vectors-than-Entries Theorem:**

> If $\{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_p\} \subset \mathbb{R}^m$ with $\boldsymbol{p} > \boldsymbol{m}$, then $\{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_p\}$ must be linearly dependent.

(2) The definition of **nonsingular matrix**.

(3) We ended the hour in midst of looking at a characterization of nonsingular matrices:

  Let $\boldsymbol{A}$ be $\boldsymbol{n} \times \boldsymbol{n}$. Then

For all $\vec{\boldsymbol{b}} \in \mathbb{R}^n$ $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a unique solution	$\Longleftrightarrow$	$\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{\theta}}$ has a unique solution

12 2/2/07 - Friday - Day Eleven

(1) We finished the proof of the nonsingular-matrix characterization theorem from Wednesday. This proof called on

  (a) The distributivity of matrix multiplication over matrix addition (which we didn't prove in class).

  (b) Linear dependence

  (c) The too-many-vectors theorem (aka the more-vectors-than-entries theorem)

13 2/5/07 - Monday - Day Twelve

(1) We began with two examples

(a) In problem 1.3: 24(a) from Assignment #4, the deal was to determine conditions on $\vec{\boldsymbol{b}}$ so that $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ is consistent (has a solution), where

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{7} & \mathbf{-1} \end{bmatrix}.$$

the text method was to work with the augmented matrix

$$\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{3} & \mathbf{-1} & \boldsymbol{b_1} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \boldsymbol{b_2} \\ \mathbf{3} & \mathbf{7} & \mathbf{-1} & \boldsymbol{b_3} \end{array}\right].$$

Another way is to work with the matrix augmented with the coefficients of the $\boldsymbol{b_i}$ thus

$$\left[\begin{array}{ccc|ccc} \mathbf{1} & \mathbf{3} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{3} & \mathbf{7} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right]$$

and do EROs until the left side is in RREF:

$$\left[\begin{array}{ccc|ccc} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{-2} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{-2} & \mathbf{1} \end{array}\right].$$

This showed the consistency criterion: $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a solution if and only if $\boldsymbol{-b_1 - 2b_2 + b_3 = 0}$. That is, $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ has a solution only for some $\vec{\boldsymbol{b}}$, not *all* $\vec{\boldsymbol{b}}$. This means that $\boldsymbol{A}$ is a singular matrix. Note that the above matrix puts us in position to write down the non-trivial solutions of $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{\theta}}.$

(b) When we treated

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} \end{bmatrix}$$

(from 1.2: 50, Assignment #3) the same way, the EROs brought us to

$$\left[\begin{array}{ccc|ccc} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1/3} & \mathbf{1} & \mathbf{-1/3} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1/6} & \mathbf{-1/2} & \mathbf{1/3} \end{array}\right].$$

The shows that $\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{b}}$ is consistent for all $\vec{\boldsymbol{b}}$, and hence that $\boldsymbol{A}$ is non-singular. The right-pocket matrix, called $\boldsymbol{B},$ here does $\boldsymbol{AB = I}.$

(2) The definition of **Matrix Inverse**. We set about to show that $\boldsymbol{B}$ above, and any matrix arising the way $\boldsymbol{B}$ did, is an inverse for $\boldsymbol{A}$ above. The definition of **invertibility.**

(3) The **Product Singular-Dominance Lemma**. It's proof had instance of decoding matrix singularity.

(4) The **Nonsingularity-Invertibility Equivalence Theorem.** We almost finished the proof.

(5) Lookahead: finish the proof. Some famous invertible matrices. And more 1.9 stuff.

14 2/6/07 - Tuesday - Day Thirteen

(1) We finished reading through the text's proof of the **Nonsingularity-Invertibility Equivalence Theorem.**

(2) This capped our work on the matrix $\boldsymbol{A}$ from 1.2: 50. We had

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{1/3} & \mathbf{1} & \mathbf{-1/3} \\ \mathbf{-1/2} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{1/6} & \mathbf{-1/2} & \mathbf{1/3} \end{bmatrix} = \frac{\mathbf{1}}{\mathbf{6}} \begin{bmatrix} \mathbf{2} & \mathbf{6} & \mathbf{-2} \\ \mathbf{-3} & \mathbf{3} & \mathbf{0} \\ \mathbf{1} & \mathbf{-3} & \mathbf{2} \end{bmatrix},$$

for which we had $\boldsymbol{AB} = \boldsymbol{I}$**,** by virtue of the way $\boldsymbol{B}$ was computed. We now also have $\boldsymbol{BA} = \boldsymbol{I}$**,** so that $\boldsymbol{B}$ is an inverse for $\boldsymbol{A}$**.**

(3) We did problem 1.9: 79 to show that if a matrix has any inverse, it has exactly one inverse. The inverse of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$**.**

(4) We ended the hour looking at Theorem 18 on things equivalent to nonsingularity.

(5) Lookahead: famous invertible matrices, the **elementary matrices.** And the intersection of almost parallel lines.

15 Assignment #5: Comment on Grading

(1) 1.7: 48

(a) It's best to work with the basic definition in the gray box on page 73. Like, directly.

(b) The hint says "Exhibit", so you could just write down a choice of a_1, a_2, and a_3 then show why it works. That is, can you tell me values for a_1, a_2, and a_3 that are guaranteed to make

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{\theta}$$

into a true equation? Can you tell me values for a_1, a_2, and a_3 which are not all of them zero, but which still guarantee that the equation is true?

(c) Watch out for the Circular-Reasoning Police. You are forbidden to begin your problem-48 solution with "Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set of vectors." That's what you want to *prove.* The Circ Police will see this as moving the finish line up to the start line.

You cannot issue such a "let" for set S because the author has already issued a "let" for S in the problem statement. You cannot change the nature of S. It's as if you were asked to write down all divisors of $n = 1728$. And you solved the problem by saying "Let $n = 1009$, then the divisors are 1 and 1009."

(2) 1.7: 49

(a) See if you can complete this nice proof by contradiction:

Suppose the assertion is not true, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ must be linearly dependent.

Thus a non-trivial linear combination of these vectors is the zero vector.

This means that we can express one of the $\vec{v}_i$ in terms of the rest. Without loss of generality, it could be $\vec{v}_3$: there are scalars x_1 and x_2 such that

$$\vec{v}_3 = x_1\vec{v}_1 + x_2\vec{v}_2$$

And so...

(b) With this set of more than two vectors, it is not enough to show that none of them is a scalar multiple of one of the others: here's a set with that property which is none-the-less linearly dependent:

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$$

(c) Here's another nice start: given the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of the problem hypothesis, let V be the matrix with these vectors as columns. Then $V^T V$ is a diagonal matrix with strictly positive entries on the main diagonal:

$$V^T V = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix},$$

where $p_i = ?$

We can render the dependence equation,

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{\theta},$$

as $V\vec{a} = \vec{\theta}$, where

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Then

$$\vec{\theta}^T\vec{\theta} = (V\vec{a})^T V\vec{a} = \cdots$$

(3) 1.7: 50

(a) How's about this? By hypothesis we have a_1, a_2, and a_3, not all zero, such that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{\theta}.$$

Let $a_4 = 0$, then we have

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{\theta},$$

where a_1, a_2, a_3, and a_4 are not all zero. Thus $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

(b) Another approach was to look at the matrix V which has the v_i, ($i = 1, 2, 3$) as columns. When we go for the RREF of $[V|\vec{\theta}]$, we get an augmented matrix which has columns indicating "free" variables. The same variables will be "free" in the solution of $[V|\vec{v}_4|\vec{\theta}]$, and so ...

16 2/7/07 - Wednesday - Day Fourteen

(1) The agenda: group work around some linear independence proofs.

(2) Click here for the sheet of problems (some typos fixed).

(3) Friday will spend time on the above sheet's 1 cdef and 2abc.

17 Current Diary

Click here for the current diary page.