This test has problems 1 – 6. Take a moment to make sure you have them all.
No Calculators Allowed; No Reference Materials; Just You and Your Pencil and Eraser.

1 Find all eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$. Be sure it’s clear which eigenvalue corresponds to which eigenvector. Show an explicit check that your eigenvectors are correct.

2 Let $E_{13}$ denote the 3-by-3 elementary matrix corresponding to the elementary operation “switch rows 1 and 3”.

(a) Evaluate the determinant of $E_{13}$.

(b) Find the inverse of $E_{13}$.

(c) Find the eigenvectors and eigenvalues for $E_{13}$.

3 Evaluate the determinant of the matrix

$$F = \begin{bmatrix} 1 & 0 & -2 & 5 \\ -2 & 1 & -5 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

4 Find an orthonormal basis for the nullspace of the 1-by-3 matrix

$$\begin{bmatrix} 3 & -9 & 6 \end{bmatrix}$$
5 Let $T$ be the function on $\mathbb{R}^3$ defined by

$$T(\vec{v}) = \vec{v} \times \vec{a},$$

where $\times$ denotes the cross product and $\vec{a} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$.

(a) Prove (directly or indirectly) that this $T$ is a linear transformation.

(b) Thinking about the geometric action of $T$, find the $\mathbb{R}^3$ eigenvectors for $T$.

(c) What eigenvalues correspond to these eigenvectors?

(d) Briefly explain whether $T$ has an invertible matrix.

6 Work at least TWO of the following parts (A)-(F). Raise your hand if you need more paper.

(A) Prove or disprove: if $\vec{z}$ is an eigenvector of $A$ corresponding to a non-zero eigenvalue, then $\vec{z}$ lies within the column space of $A$.

(B) Write a proof of the theorem that lies at the basis of the idea of dimension: Let $W$ be a subspace of $\mathbb{R}^Q$, and suppose that $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is a spanning set for $W$. Let $\{\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_m\}$ be a subset of $W$. If $m > n$, then $\{\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_m\}$ must be linearly dependent.

(C) If you have done part (B), explain why the theorem therein makes the dimension of $W$ a well-defined idea.

(D) Give a proof of the theorem about how the reduced echelon form of matrix $A$ indicates a column-space basis made up of actual columns of $A$.

(E) Suppose $M$ is an invertible matrix. Briefly explain the relationship between the eigenvalues and eigenvectors of $M$ and the eigenvalues and eigenvectors of $M^{-1}$.

(F) Give a proof of the theorem that says matrix $A$ is nonsingular if and only if $A$ can be expressed as a product of elementary matrices.