The take-home test part here requires that you do problem **7** and then four (**4**) of the parts of problem 8. Note that problem 8 is not quite the same here as it was on the in-class portion of test #2.

7 By explicit appeal to the most basic definitions, prove that $\boldsymbol{H}$, given by

$$\boldsymbol{H} = \left\{\vec{\boldsymbol{x}} \;:\; \vec{\boldsymbol{x}} \in \boldsymbol{R}^{\mathbf{4}} \text{ and } \boldsymbol{x}_{\mathbf{2}} = \mathbf{0}\right\},$$

is a subspace of $\boldsymbol{R}^{\mathbf{4}}$. (15 points)

8 For each chosen one of the following parts (A)-(F), provide

(i) a proof, or

(ii) a proof of the negation, or

(iii) a counterexample.

A counterexample must be accompanied by a brief explanation, with necessary computations, as to why the counterexample does the job.

An example which verifies the assertion is **not** it. Proofs must be general.

(A) Let $\boldsymbol{A}$ be an arbitrary $\boldsymbol{m} \times \boldsymbol{n}$ matrix. If $\left\{\vec{\boldsymbol{u}}_{\mathbf{1}}, \ldots, \vec{\boldsymbol{u}}_{\boldsymbol{p}}\right\}$ is linearly independent, then $\left\{\boldsymbol{A}\vec{\boldsymbol{u}}_{\mathbf{1}}, \ldots, \boldsymbol{A}\vec{\boldsymbol{u}}_{\boldsymbol{p}}\right\}$ is also linearly independent.

(B) Let matrix $\boldsymbol{C}$ be given by

$$\boldsymbol{C} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \end{bmatrix}.$$

Define the function $\boldsymbol{f}$: for each $\vec{\boldsymbol{x}}$ in $\boldsymbol{R}^{\mathbf{3}}$, let

$$\boldsymbol{f}(\vec{\boldsymbol{x}}) = \mathbf{det}\big(\boldsymbol{C}_{\mathbf{2}}(\vec{\boldsymbol{x}})\big),$$

that is, $\boldsymbol{f}(\vec{\boldsymbol{x}})$ is the determinant of the matrix obtained from $\boldsymbol{C}$ by replacing column 2 with $\vec{\boldsymbol{x}}$.

Appealing to determinant theorems, prove that $\boldsymbol{f}$ is actually a linear transformation. Find the standard matrix of $\boldsymbol{f}$.

(C) In $\boldsymbol{R^3}$, we can define the famous cross product of two vectors in the following fashion:

$$\vec{x} \times \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\ -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \\ \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \end{bmatrix},$$

where the product's entries are the $\mathbf{2 \times 2}$ determinants shown.

It's easy to come up with the product's first entry: just cover up the operands' first entries and use the determinant of the still-visible entries. For the product's second entry, cover the operands' second entries and use *minus* the value of the determinant of the remaining operand entries. Then the product's third entry...

Examples:

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \qquad \vec{e}_2 \times \vec{e}_1 = -\vec{e}_3, \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} -7 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -26 \\ 16 \end{bmatrix}$$

Let $\vec{a}$ be a fixed vector in $\boldsymbol{R^3}$. Define the function $f : \boldsymbol{R^3} \longrightarrow \boldsymbol{R^3}$ by $f(\vec{x}) = \vec{a} \times \vec{x}$. Prove that f is a *linear transformation.* And find the standard matrix for f is terms of the entries of the fixed vector $\vec{a}$.

(D) Let the function $T : \boldsymbol{R^n} \longrightarrow \boldsymbol{R^n}$ be defined as follows: to find $T(\vec{x})$, sort the entries of $\vec{x}$ into descending order. For example, in $\boldsymbol{R^3}$,

$$T\left(\begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}, \quad T\left(\begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}.$$

Show T is a *linear transformation.*

(E) Let $\boldsymbol{V}$ be the vector space of all $\mathbf{2} \times \mathbf{2}$ matrices. Let $\boldsymbol{H}$ be the set of all matrices in $\boldsymbol{V}$ whose determinant is zero. Prove that $\boldsymbol{H}$ is a *subspace* of $\boldsymbol{V}$. Determine a basis for $\boldsymbol{H}$.

(F) Let $\left\{\vec{\boldsymbol{u}}_1, \ldots, \vec{\boldsymbol{u}}_5\right\}$ be a linearly-independent subset of vector space $\boldsymbol{V}$. Let $\left\{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_3\right\}$ be given by

$$\begin{aligned}
\vec{\boldsymbol{w}}_1 &= -5\vec{\boldsymbol{u}}_1 + 10\vec{\boldsymbol{u}}_2 + 15\vec{\boldsymbol{u}}_3 \\
\vec{\boldsymbol{w}}_2 &= 3\vec{\boldsymbol{u}}_1 - 8\vec{\boldsymbol{u}}_2 + \vec{\boldsymbol{u}}_3 \\
\vec{\boldsymbol{w}}_3 &= 4\vec{\boldsymbol{u}}_1 - 9\vec{\boldsymbol{u}}_2 + 2\vec{\boldsymbol{u}}_3.
\end{aligned}$$

Show that $\left\{\vec{\boldsymbol{w}}_1, \ldots, \vec{\boldsymbol{w}}_3\right\}$ is a *linearly independent* subset of $\boldsymbol{V}$ by direct appeal to the most basic definition.