In this long example we hunt for the zeros of \( P \), where
\[
P(x) = 4x^5 + 16x^4 + 9x^3 - 25x^2 - 16x + 12
\]
During this hunt, we use

(i) Rational-Zeros List (BLUE 279)

(ii) Synthetic Division (3.2)

(iii) The Reduced Polynomial.

(iv) Descartes’s Rules of Signs (First BLUE283)

(v) Bounds on Real Zeros (Second BLUE 283)

1. Rational-Zeros List: we see that the constant term, 12, has divisors
\[
\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12
\]
while the leading coefficient, 4, has only
\[
\pm 1, \pm 2, \pm 4
\]
as divisors. The rational numbers which could possibly be zeros are then
\[
\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}
\]

2. We begin by checking whether \( x = 1 \) yields a zero:

\[
\begin{array}{cccccc}
1 & 4 & 16 & 9 & -25 & -16 & 12 \\
& 4 & 20 & 29 & 4 & -12 & 0 \\
\end{array}
\]
The lowest right-most entry is zero. This tells us that \( P(1) = 0 \), that is, \( x = 1 \) is a zero of \( P \).
When you’ve found a zero, as we have here with \( x = 1 \), it’s always worth trying it again. It may be of \textit{multiplicity} more than 1. We try \( x = 1 \) in the \textit{reduced polynomial},

\[
P_1(x) = 4x^4 + 20x^3 + 29x^2 + 4x - 12
\]

(Note that \( P(x) = (x - 1)P_1(x) \) with

\[
\begin{array}{cccccc}
1 & 4 & 20 & 29 & 4 & -12 \\
 & 4 & 24 & 53 & 57 & \\
4 & 24 & 53 & 57 & 45 & \\
\end{array}
\]

The 45 in the remainder (or \( P_1(1) \)) spot tells us \( x = 1 \) does not furnish us a zero of \( P_1 \).

So we know that \( x = 1 \) is only a multiplicity-one zero of \( P \).

Let’s try \( x = -1 \) in the reduced polynomial:

\[
\begin{array}{cccccc}
-1 & 4 & 20 & 29 & 4 & -12 \\
 & -4 & -16 & -13 & 9 & \\
4 & 16 & 13 & -9 & -3 & \\
\end{array}
\]

The -3 in the lower right-hand position shows that \( x = -1 \) is not a zero of \( P \).

Let’s try \( x = 2 \) in the reduced polynomial:

\[
\begin{array}{cccccc}
2 & 4 & 20 & 29 & 4 & -12 \\
 & 8 & 56 & 170 & 348 & \\
4 & 28 & 85 & 174 & 336 & \\
\end{array}
\]

The 336 in the lower right-hand position shows that \( x = 2 \) is not a zero of \( P \).

This try reminds us of the Upper Bound on the real zeros: \( P \) can have no zeros greater than \( x = 2 \). This means that we won’t try any of the larger rational candidates.

Let’s try \( x = -2 \) in the reduced polynomial:

\[
\begin{array}{cccccc}
-2 & 4 & 20 & 29 & 4 & -12 \\
 & -8 & -24 & -10 & 12 & \\
4 & 12 & 5 & -6 & 0 & \\
\end{array}
\]

The 0 in the lower right-hand position shows that \( x = -2 \) is a zero of \( P \).
7 Let’s re-try \( x = -2 \), using our new reduced polynomial \( P_2 \),
\[
P_2(x) = 4x^3 + 12x^2 + 5x - 6,
\]
this time
\[
\begin{array}{cccc}
-2 & 4 & 12 & 5 & -6 \\
& -8 & -8 & 6 \\
& 4 & 4 & -3 & 0 \\
\end{array}
\]
This shows that \( x = -2 \) is a multiplicity-two zero of \( P \), and that, so far, we have
\[
P(x) = (x - 1)(x + 2)^2(4x^2 + 4x - 3).
\]

8 Our new reduced polynomial \( 4x^2 + 4x - 3 \) is quadratic, and
\[
4x^2 + 4x - 3 = (2x - 1)(2x + 3),
\]
which has zeros \( \frac{1}{2} \) and \( -\frac{3}{2} \).

9 Thus, our original polynomial \( P(x) \) has five real zeros:
\[
1, -2, -2, \frac{1}{2}, -\frac{3}{2}.
\]

There is also the factorization:
\[
P(x) = (x - 1)(x + 2)^2(2x - 1)(2x + 3).
\]