Refer to sections 8.3 and 8.4 of our text.

Here we demonstrate a way of using the celebrated **Gauss-Jordan Elimination Method** and the **Reduced Row-Echelon Form** (page 655, example 6) to solve the system of equations: $\boldsymbol{2x+5y+4z=4}$, $\boldsymbol{x+4y+3z=1}$, and $\boldsymbol{x-3y-2z=5}$. It would be easier to write the equations in other than the following order. But not as instructive.

$$\begin{aligned} \boldsymbol{2x+5y+4z} &= \boldsymbol{4} \\ \boldsymbol{x+4y+3z} &= \boldsymbol{1} \\ \boldsymbol{x-3y-2z} &= \boldsymbol{5} \end{aligned}$$

We begin by racking up the equations and forming the *augmented matrix*:

$$\begin{aligned} \boldsymbol{2x+5y+4z} &= \boldsymbol{4} \\ \boldsymbol{x+4y+3z} &= \boldsymbol{1} \\ \boldsymbol{x-3y-2z} &= \boldsymbol{5} \end{aligned} \qquad \left[\begin{array}{rrr|r} \boldsymbol{2} & \boldsymbol{5} & \boldsymbol{4} & \boldsymbol{4} \\ \boldsymbol{1} & \boldsymbol{4} & \boldsymbol{3} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{-3} & \boldsymbol{-2} & \boldsymbol{5} \end{array}\right]_{(\boldsymbol{I})}$$

We move on to the second GEM matrix by multiplying the first row/equation through by $\boldsymbol{1/2}$. This is one of the three **Elementary Operations** (see BLUE639 and BLUE649). The resulting matrix begins:

$$\left[\begin{array}{rrr|r} \boldsymbol{1} & \boldsymbol{\frac{5}{2}} & \boldsymbol{2} & \boldsymbol{2} \\ & & & \\ & & & \end{array}\right]_{(\boldsymbol{II})}$$

The remaining rows of this second matrix arise from adding multiples of this new first row to the previous second and third rows (this is a repeated invocation of the first of the elementary operations). To keep track of all this, the operator arranges the multiples of the new first row to look like exponents. He goes back and messes up the previous augmented matrix:

$$\left[\begin{array}{rrr|r} \boldsymbol{2} & \boldsymbol{5} & \boldsymbol{4} & \boldsymbol{4} \\ \boldsymbol{1^{-1}} & \boldsymbol{4^{-5/2}} & \boldsymbol{3^{-2}} & \boldsymbol{1^{-2}} \\ \boldsymbol{1^{-1}} & \boldsymbol{-3^{-5/2}} & \boldsymbol{-2^{-2}} & \boldsymbol{5^{-2}} \end{array}\right]_{(\boldsymbol{I})} \quad \begin{aligned} \boldsymbol{N_1} &= \boldsymbol{O_1/2} \\ \boldsymbol{N_2} &= \boldsymbol{O_2+(-1)N_1} \\ \boldsymbol{N_3} &= \boldsymbol{O_3+(-1)N_1} \end{aligned} \quad \left[\begin{array}{rrr|r} \boldsymbol{1} & \boldsymbol{\frac{5}{2}} & \boldsymbol{2} & \boldsymbol{2} \\ \boldsymbol{0} & \boldsymbol{\frac{3}{2}} & \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{0} & \boldsymbol{-\frac{11}{2}} & \boldsymbol{-4} & \boldsymbol{3} \end{array}\right]_{(\boldsymbol{II})}$$

Here the operator got the second and third rows of $(\boldsymbol{II})$ by adding the little exponent-ish numbers in $(\boldsymbol{I})$ to their “bases”.

Now to do the $(\boldsymbol{II})$-to-$(\boldsymbol{III})$ transition:

$$\left[\begin{array}{ccc|c} & & & \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ & & & \end{array}\right]_{(III)}$$

Now to go back and decorate (II) with the little exponent-ish numbers which come from multiplying this new second row by non-zero constants:

$$\left[\begin{array}{ccc|c} 1^{0} & \frac{5}{2}^{-5/2} & 2^{-10/6} & 2^{10/6} \\ 0 & \frac{3}{2} & 1 & -1 \\ 0^{0} & -\frac{11}{2}^{11/2} & -4^{11/3} & 3^{-11/3} \end{array}\right]_{(II)} \quad \begin{aligned} N_2 &= (3/2)O_2 \\ N_1 &= O_1 + (-5/2)N_2 \\ N_3 &= O_3 + (11/2)N_2 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{11}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right]_{(III)}$$

Now for the (III)-to-(IV) transition:

$$\left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 1 & 2 \end{array}\right]_{(IV)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3}^{-1/3} & \frac{11}{3}^{-2/3} \\ 0 & 1 & \frac{2}{3}^{-2/3} & -\frac{2}{3}^{-4/3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right]_{(III)} \quad \begin{aligned} N_3 &= (-3)O_3 \\ N_1 &= O_1 + (-1/3)N_3 \\ N_2 &= O_2 + (-2/3)N_3 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]_{(IV)}$$

This last augmented matrix tells us that the solution is $x = 3$, $y = -2$, $z = 2$.

The above system of equations had just one solution, the one we found. To cap off our knowledge of systems of linear equations we need to look at systems not of the just-one-solution type.

$$\begin{aligned} x + 2y + 3z &= 17 \\ 4x + 5y + 6z &= 47 \\ 7x + 8y + 9z &= 77 \end{aligned} \qquad \begin{aligned} x + 2y + 3z &= 5 \\ 4x + 5y + 6z &= 1 \\ 7x + 8y + 9z &= 13 \end{aligned} \qquad \begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 9z &= 0 \end{aligned}$$

The leftmost system engenders the following sequence of augmented matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 17 \\ 4 & 5 & 6 & 47 \\ 7 & 8 & 9 & 77 \end{array}\right]_{(I)} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 17 \\ 4^{-4} & 5^{-8} & 6^{-12} & 47^{-68} \\ 7^{-7} & 8^{-14} & 9^{-21} & 77^{-119} \end{array}\right]_{(I)} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 17 \\ 0 & -3 & -6 & -21 \\ 0 & -6 & -12 & -42 \end{array}\right]_{(II)}$$

$$\left[\begin{array}{ccc|c} 1 & 2^{-2} & 3^{-4} & 17^{-14} \\ 0 & -3 & -6 & -21 \\ 0 & -6^{6} & -12^{-12} & -42^{42} \end{array}\right]_{(II)} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]_{(III)} \quad \begin{aligned} x + 0y - z &= 3 \\ 0x + y + 2z &= 7 \\ 0x + 0y + 0z &= 0 \end{aligned}$$

We see that *any* values of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ will satisfy the third equation (the one corresponding to the row of zeros all the way across RREF $(\boldsymbol{III})$). This is GEM's way of telling us that we could have gotten the same effect using only two of the initial three equations. Not just any two. But which two, it doesn't say.

The remaining two equations allow us to express some of the variables in terms of the others, as we will now see.

This is traditionally done so that the right-most possible variables are "independent": here $\boldsymbol{z}$ is right-most; we make it "independent"; we make $\boldsymbol{x}$ and $\boldsymbol{y}$ depend on $\boldsymbol{z}$:

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{z+3} \\ \boldsymbol{y} &= \boldsymbol{-2z+7} \end{aligned}$$

This gives us a formula for the infinitely many solutions of this system: for every value of $\boldsymbol{z}$ we choose, we get a solution:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$
$\mathbf{3}$	$\mathbf{7}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{13}$	$\mathbf{-3}$
$\mathbf{13/2}$	$\mathbf{0}$	$\mathbf{7/2}$
$\mathbf{4}$	$\mathbf{5}$	$\mathbf{1}$
$\mathbf{5}$	$\mathbf{3}$	$\mathbf{2}$
$\mathbf{6}$	$\mathbf{1}$	$\mathbf{3}$
$\mathbf{7}$	$\mathbf{-1}$	$\mathbf{4}$

A further tradition involves writing the variables in terms of new variables: letting $\boldsymbol{z=t}$, we have

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{t+3} \\ \boldsymbol{y} &= \boldsymbol{-2t+7} \\ \boldsymbol{z} &= \boldsymbol{t} \end{aligned}$$

This emphasizes that the system has as many solutions as there are values of $\boldsymbol{t}$.

Here are some nice-numbers systems which you should be able to do by hand:

1 One solution: $\boldsymbol{x = 1/2}$, $\boldsymbol{y = -3}$, $\boldsymbol{z = 2/3}$:

$$\begin{aligned} 2x - 2y + 6z &= 11 \\ 4x + y + 3z &= 1 \\ -6x + 3y + 12z &= -4 \end{aligned}$$

2 This system turns out to have no solutions (our text calls this **inconsistent**, see page BLUE654):

$$\begin{aligned} x + 2y + 4z &= 6 \\ y + 2z &= 3 \\ x + y + 2z &= 1 \end{aligned}$$

The indicator for this is a row of form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & K \end{array}\right]$$

where $\boldsymbol{K \neq 0}$. Do you see why this is a no-solutions situation?

3 An infinitely-many-solutions example (some folks call this **dependent**, also in BLUE654):

$$\begin{aligned} x + 2y + 4z &= 9 \\ y + 2z &= 3 \\ x + y + 2z &= 6 \end{aligned}$$

The solution set can be expressed as

$$\begin{aligned} x &= 3 \\ y &= -2z + 3 \end{aligned}$$

or

$$\begin{aligned} x &= 3 \\ y &= -2t + 3 \\ z &= t \end{aligned}$$

4 An "overdetermined" system. Read the story, dream up four relevant variables, write a system of linear equations governing the variables, write the sytem's augmented matrix, coerce to reduced row-echelon form, announce the system's solution set, announce the problem's solution.

Here's the story:

> A number has four non-zero digits: the first digit is five times the last, the second is four more than the first and three times the third and the third is two more than the last and two less than the first. Find the number.