

**Solutions Homework Assignment 7, MATH 515, Spring 09**

**Problem 17)** (a) From the definition  $(g \circ f)_*(\mathcal{M}) = \{W \subset Z : (g \circ f)^{-1} = f^{-1}(g^{-1}(W)) \in \mathcal{M}\}$ , and  $g_*(f_*(\mathcal{M})) = \{W \subset Z : g^{-1}(W) \in f_*(\mathcal{M})\}$ , where  $f_*(\mathcal{M}) = \{V \subset Y : f^{-1}(V) \in \mathcal{M}\}$ . Thus  $W \in (g \circ f)_*(\mathcal{M}) \iff f^{-1}(g^{-1}(W)) \in \mathcal{M} \iff g^{-1}(W) \in f_*(\mathcal{M}) \iff W \in g_*(f_*(\mathcal{M}))$ .

(b) For  $B \in f_*(\mathcal{M})$  we have  $f_*(\mu)(B) = \mu(f^{-1}(B)) \geq 0$ , and  $f_*(\mu)(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$ . Let  $\{B_i\}$  be a denumerable family of disjoint subsets of  $Y$  with  $B_i \in f_*(\mathcal{M}) \iff f^{-1}(B_i) \in \mathcal{M}$  for  $i \in \mathbb{N}$ . Then  $\{f^{-1}(B_i)\}$  is a denumerable collection of disjoint subsets of  $X$ , and

$$\begin{aligned} f_*(\mu)\left(\bigcup_{i=1}^{\infty} B_i\right) &= \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right) \\ &= \sum_{i=1}^{\infty} \mu(f^{-1}(B_i)) = \sum_{i=1}^{\infty} (f_*(\mu)(B_i)) \end{aligned}$$

**Problem 18)** Consider  $f_n : X \rightarrow \mathbb{R}$  measurable for  $n$  positive integers. Then for all positive integers  $k, m, n$  we have that the sets

$$D(k, n, m) := |f_n - f_m|^{-1}\left[\frac{1}{k}, \infty\right) = \{x \in X : |f_n - f_m| \geq \frac{1}{k}\}$$

are measurable because  $[\frac{1}{k}, \infty)$  is a Borel set and  $|f_n - f_m|$  is a measurable function. Thus also

$$D := \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} D(k, n, m)$$

is measurable by the definition of  $\sigma$ -algebra. But  $x \in D$  if and only if the sequence of real numbers  $\{f_n(x)\}$  is not a Cauchy sequence, which is equivalent to  $\{f_n(x)\}$  is not convergent. But then  $S = X - D$  is measurable too.

**Problem 19)** For the proofs of (a) and (b) refer to Rudin, 3.17. The only difference is that the sequences can actually take values  $\pm\infty$ . So for example if we assume that there  $a_n > b + \varepsilon$  for infinitely many  $n$  then either for infinitely many  $n$ ,  $a_n \in [b + \varepsilon, c]$  and by compactness there is an accumulation point  $b' > b$ , or there are for each number  $c$  infinitely many  $n$  with  $a_n > c$ , in which case  $\infty$  is an accumulation point.  $\limsup(a_n) = \infty$  if and only if for each number  $c$ , for infinitely many  $n$ ,  $a_n \in (c, \infty]$ . We have  $\liminf(a_n) = \infty$  if and only if for each number  $c$ ,  $a_n \in (c, \infty]$  for all but finitely many  $n$ . Similarly we have  $\limsup(a_n) = -\infty$  if for each number  $c$  for all but finitely many  $n$  we have  $a_n < c$ .

(c) This is Examples 3.18 (c) in Rudin for real sequences. The argument is very similar. If  $a = \liminf(a_n) = \limsup(a_n)$  is a number then from the characterization in (b) we know that for only finitely many  $n$ ,  $a_n \geq a + \varepsilon$  and for only finitely many  $n$ ,  $a_n \leq a - \varepsilon$ , thus  $a_n \in (a - \varepsilon, a + \varepsilon)$  for all but finitely many  $n$ ,

which means  $a_n$  converges to  $a$ . The cases  $a = \pm\infty$  are similar because of the definitions above.

(d) It follows from the characterization in (b) that  $f(x) = \limsup(f_n(x))$  respectively  $g(x) = \liminf(f_n(x))$  are accumulation points of  $\{f_n(x)\}$  for each  $x \in X$ . Thus there exists a subsequence of  $\{f_n(x)\}$  converging to  $f(x)$ . But the pointwise limit of a sequence of measurable functions is measurable. This proves the claim.

Note that  $\limsup(a_n)$  is also characterized by  $\lim_{N \rightarrow \infty}(\sup_{n \geq N}(a_n))$ , see e. g. Ross, Elementary Analysis, page 57. This is a workable definition too, and the equivalence is immediate from (b) above. This definition of  $\limsup$  also immediately implies measurability. In fact  $\sup_{n \geq N}(f_n) = \lim_{m \rightarrow \infty} \sup\{f_N, \dots, f_{N+m}\}$  is measurable because the supremum of a finite number of measurable functions is measurable. Then the rest follows because limits of measurable functions are measurable.