Problem 14) Let $B'$ be a Hamel basis of $E$. Because $B'$ is not finite we can choose a sequence of basis elements $\{v_i\}$ in $B'$. Then we can change $B'$ to a Hamel basis $B$ by replacing $v_i$ by $\frac{w_i}{\|w_i\|}$. Note that $v_i \neq 0$ because $B$ is linearly independent. (Otherwise $1 \cdot v_i = 0 \cdot v_i$ would be two different representations of the 0-vector in $E$.) Now define $\lambda : E \to \mathbb{R}$ (respectively $\mathbb{C}$) by $\lambda(v_i) = i$ and $\lambda(v) = 0$ for $v \in B \setminus \{v_i\}$. Note that this defines a unique linear map $\lambda : E \to \mathbb{R}$ because each $x \in E$ has a unique representation $x = \sum_{finite} c_i b_i$ with $b_i \in B$ so that $\lambda(\sum c_i b_i) = \sum c_i \lambda(b_i)$. (If $\lambda$ is defined in this way it is linear because if $y = \sum d_i b_i$, where we can assume the representation is given with the same finite number of elements $b_i$ from the Hamel basis as for $x$ because we can set $c_i$ respectively $d_i$ equal to 0 for the corresponding basis elements if not. Then $\lambda(x + y) = \sum (c_i + d_i) \lambda(b_i) = \sum c_i \lambda(b_i) + \sum d_i \lambda(b_i) = \lambda(x) + \lambda(y)$. Also $\lambda(c x) = \lambda(\sum c c_i b_i) = \sum c c_i \lambda(b_i) = c \sum c_i \lambda(b_i) = c\lambda(x)$ for all scalars $c$.) Note that
\[
\sup_{S(E)} |\lambda(b)| = \infty
\]
because there is the sequence $v_i$ in $S(E)$ with $|\lambda(i)| = i \to \infty$ for $i \to \infty$.

Problem 15) Given $\lambda$ we know that
\[
|\lambda| = \sup_{x \in S(E)} |\lambda(x)|,
\]
and so if $|\lambda| = 1$ we know that we can find for each $\varepsilon > 0$ some $x_\varepsilon \in S(E)$ such that $|\lambda(x_\varepsilon)| > 1 - \varepsilon$. But $x_\varepsilon \in S(E) \iff |x_\varepsilon| = 1$. Now for each $\varepsilon > 0$ we have $|\lambda(x_\varepsilon)| = |\lambda(x_\varepsilon) e^{i\theta}|$ where $\theta$ can depend on $\varepsilon$. Then $\lambda(x_\varepsilon e^{i\theta}) = e^{i\theta} \lambda(x_\varepsilon) = |\lambda(x_\varepsilon)|$ and $|x_\varepsilon e^{i\theta}| = |x_\varepsilon| = 1$. (In the real case it suffices to replace $x_\varepsilon$ by $-x_\varepsilon$ if $\lambda(x_\varepsilon) < 0$.) For the counter example consider $E = \ell_1$ and define $\lambda : E \to \mathbb{R}$ by $\lambda(x) = \sum_{i=1}^{\infty} x_i (1 - \frac{1}{i})$ for $x = \{x_i\} \in \ell_1$. This is well-defined because $\sum_{i=1}^{\infty} |x_i| (1 - \frac{1}{i}) \leq \sum_{i=1}^{\infty} |x_i| < \infty$ for all $x \in E$, and absolute convergence implies convergence of a series of complex numbers. Moreover, $\lambda$ is linear because $\lambda(x+y) = \sum_{i=1}^{\infty} (x_i+y_i) (1 - \frac{1}{i}) = \sum_{i=1}^{\infty} x_i (1 - \frac{1}{i}) + \sum_{i=1}^{\infty} y_i (1 - \frac{1}{i}) = \lambda(x) + \lambda(y)$, and $\lambda(c x) = \sum_{i=1}^{\infty} (c x_i) (1 - \frac{1}{i}) = c \sum_{i=1}^{\infty} x_i (1 - \frac{1}{i}) = c x \lambda(x)$, for $c \in \mathbb{C}$ and $x, y \in E$. Now, by the estimate above, $\lambda$ is bounded and $|\lambda| = \sup_{x \in S(E)} |\lambda(x)| = 1$ because for the sequence $e_i \in S(E) \subset E$ defined by $e_i := \{e_{ij}\}$ defined by $e_{ij} = 0$ if $i \neq j$ and $e_{ii} = 1$ we have $\lambda(e_i) = (1 - \frac{1}{i}) \to 1$ for $i \to \infty$. Moreover, if $x \in S(E)$ then there exists a smallest index $j$ such that $x_j \neq 0$. Then $\lambda(x) = x_j (1 - \frac{1}{j}) + \sum_{i \neq j} x_i(1 - \frac{1}{i})$ and thus $|\lambda(x)| \leq |x_j| (1 - \frac{1}{j}) + \sum_{i \neq j} |x_i| < \sum_{i=1}^{\infty} |x_i| = |x| = 1$. (The reference to $\ell_\infty$ comes from the fact that the above functional is a special case (with $y = \{1 - \frac{1}{i}\}$) of the pairing
\[
\ell_1 \times \ell_\infty \to \mathbb{C}
\]
defined by $(x, y) \mapsto \sum_{i=1}^{\infty} x_i y_i$, which is well-defined because $\sum_{i=1}^{\infty} |x_i y_i| \leq |x_i| |y_i|_{\infty}$. This bilinear pairing induces isometric linear maps $\ell_1 \to \ell_\infty$ and $\ell_\infty \to$.
The first induces an isometry between $\ell_1$ and $c_0$, where $c_0$ is the subspace of null-sequences in $\ell_\infty$ while the second one is an isometric isomorphism. It is also known that $\ell'_\infty$ is not isometric with $\ell_1$.

**Problem 16** (a) If $F$ is a finite dimensional subspace of a normed vector space and $\{x_n\}$ is a sequence in $F$, which converges to a limit in $E$ then it is a Cauchy sequence in $F$. Because $F$ (with any norm) is complete this Cauchy sequence converges to a limit in $F$. Thus $F$ is closed.

(b) Let $K = \mathbb{R}$ respectively $\mathbb{C}$ depending on whether we consider real or complex spaces. Note that $F$, being a finite dimensional vector space, is isomorphic to $K^n$ by the choice of a basis $\{v_i\}_{i=1,...,n}$. Consider the identity map $I : F \to F$. The basis defines continuous functionals $\lambda_i : F \to K$ by $\lambda_i(\sum_{i=1}^{n} c_i v_i) = c_i$ for $i = 1, \ldots, n$. (The continuity is for example obvious with the supremum norm but does not depend on the choice of norm in the finite dimensional setting.) Now extend each $\lambda_i$ by the Hahn-Banach theorem to continuous functionals $\Lambda_i : E \to K$ and consider the continuous linear map $P : E \to F$ defined by $P(v) = \sum_{i=1}^{n} \Lambda_i(v) v_i$. **Claim:** $G := \ker(P) = \{ v \in E : P(v) = 0 \}$ is a closed subspace of $E$ such that $E = F + G$ and $F \cap G = \{0\}$. **Proof.** $\ker(P)$ is obviously a subspace, which is closed because $\ker(P) = P^{-1}(\{0\})$ and $\{0\} \subset F$ is a closed subset. For each $v \in E, v - P(v) \in G$ because $P(v - P(v)) = P(v) - P(P(v)) = 0$ because $P(v) \in F$ and $P|F = I$ and thus $P(P(v)) = P(v)$. Thus $v - P(v) = w \in G$ and $v = P(v) + w$. This shows $E = F + G$. Moreover, if $v \in F \cap G$ then $P(v) = v$ and $P(v) = 0$ and thus $v = 0$. This shows $F \cap G = \{0\}$. 
