Solutions Homework Assignment 6, MATH 515, Spring 09

Problem 14) Let $\mathcal{B}'$ be a Hamel basis of E. Because $\mathcal{B}'$ is not finite we can choose a sequence of basis elements $\{v_i\}$ in $\mathcal{B}'$. Then we can change $\mathcal{B}'$ to a Hamel basis $\mathcal{B}$ by replacing v_i by $\frac{v_i}{|v_i|}$. Note that $v_i \neq 0$ because $\mathcal{B}$ is linearly independent. (Otherwise $1 \cdot v_i = 0 \cdot v_i$ would be two different representations of the 0-vector in E.). Now define $\lambda : E \to \mathbb{R}$ (respectively $\mathbb{C}$) by $\lambda(v_i) = i$ and $\lambda(v) = 0$ for $v \in \mathcal{B} \setminus \{v_i\}$. Note that this defines a unique linear map $\lambda : E \to \mathbb{R}$ because each $x \in E$ has a *unique* representation $x = \sum_{\text{finite}} c_b b$ with $b \in \mathcal{B}$ so that $\lambda(\sum c_b b) = \sum c_b \lambda(b)$. (If λ is defined in this way it is linear because if $y = \sum d_b b$, where we can assume the representation is given with the same finite number of elements b from the Hamel basis as for x because we can set c_b respectively d_b equal to 0 for the corresponding basis elements if not. Then $\lambda(x+y) = \sum (c_b + d_b)\lambda(b) = \sum c_b \lambda(b) + \sum d_b \lambda(b) = \lambda(x) + \lambda(y)$. Also $\lambda(cx) = \lambda(\sum c c_b b) = \sum c c_b \lambda(b) = c \sum c_b \lambda(b) = c\lambda(x)$ for all scalars c.) Note that

$$\sup_{S(E)} |\lambda(b)| = \infty$$

because there is the sequence v_i in $S(E)$ with $|\lambda(i) = i \to \infty$ for $i \to \infty$.

Problem 15) Given λ we know that

$$|\lambda| = \sup_{x \in S(E)} |\lambda(x)|,$$

and so if $|\lambda| = 1$ we know that we can find for each $\varepsilon > 0$ some $x_\varepsilon \in S(E)$ such that $|\lambda(x_\varepsilon)| > 1 - \varepsilon$. But $x_\varepsilon \in S(E) \iff |x_\varepsilon| = 1$. Now for each $\varepsilon > 0$ we have $|\lambda(x_\varepsilon)| = \lambda(x_\varepsilon)e^{i\theta}$ where θ can depend on ε. Then $\lambda(x_\varepsilon e^{i\theta}) = e^{i\theta}\lambda(x_\varepsilon) = |\lambda(x_\varepsilon)|$ and $|x_\varepsilon e^{i\theta}| = |x_\varepsilon| = 1$. (In the real case it suffices to replace x_ε by $-x_\varepsilon$ if $\lambda(x_\varepsilon) < 0$.) For the counter example consider $E = \ell_1$ and define $\lambda : E \to \mathbb{R}$ by $\lambda(x) = \sum_{i=1}^\infty x_i(1 - \frac{1}{i})$ for $x = \{x_i\} \in \ell_1$. This is well-defined because $\sum_{i=1}^\infty |x_i||1 - \frac{1}{i}| \leq \sum_{i=1}^\infty |x_i| < \infty$ for all $x \in E$, and absolute convergence implies convergence of a series of complex numbers. Moreover, λ is linear because $\lambda(x+y) = \sum_{i=1}^\infty (x_i + y_i)(1 - \frac{1}{i}) = \sum_{i=1}^\infty x_i(1 - \frac{1}{i}) + \sum_{i=1}^\infty y_i(1 - \frac{1}{i}) = \lambda(x) + \lambda(y)$, and $\lambda(cx) = \sum_{i=1}^\infty (cx_i)(1 - \frac{1}{i}) = c\sum_{i=1}^\infty x_i(1 - \frac{1}{i}) = x\lambda(x)$, for $c \in \mathbb{C}$ and $x, y \in E$. Now, by the estimate above, λ is bounded and $|\lambda| = \sup_{x \in S(E)} |\lambda(x)| = 1$ because for the sequence $e_i \in S(E) \subset E$ defined by $e_i := \{e_{ij}\}$ defined by $e_{ij} = 0$ if $i \neq j$ and $e_{ii} = 1$ we have $\lambda(e_i) = (1 - \frac{1}{i}) \to 1$ for $i \to \infty$. Moreover, if $x \in S(E)$ then there exists a smallest index j such that $x_j \neq 0$. Then $\lambda(x) = x_j(1 - \frac{1}{j}) + \sum_{i>j} x_i(1 - \frac{1}{i})$ and thus $|\lambda(x)| \leq |x_i|(1 - \frac{1}{j}) + \sum_{i>j} |x_j| < \sum_{i=1}^\infty |x_i| = |x| = 1$. (The reference to ℓ_∞ comes from the fact that the above functional is a special case (with $y = \{1 - \frac{1}{i}\}$) of the pairing

$$\ell_1 \times \ell_\infty \to \mathbb{C}$$

defined by $(x, y) \mapsto \sum_{i=1}^\infty x_i y_i$, which is well-defined because $\sum_{i=1}^\infty |x_i y_i| \leq |x|_1 |y|_\infty$. This bilinear pairing induces isometric linear maps $\ell_1 \to \ell_\infty'$ and $\ell_\infty \to$

ℓ_1'. The first induces an isometry between ℓ_1 and c_0', where c_0 is the subspace of null-sequences in ℓ_∞ while the second one is an isometric isomorphism. It is also known that ℓ_∞' is not isometric with ℓ_1.)

Problem 16) (a) If F is a finite dimensional subspace of a normed vector space and $\{x_n\}$ is a sequence in F, which converges to a limit in E then it is a Cauchy sequence in F. Because F (with any norm) is complete this Cauchy sequence converges to a limit in F. Thus F is closed.

(b) Let $\mathbb{K} = \mathbb{R}$ respectively $\mathbb{C}$ depending on whether we consider real or complex spaces. Note that F, being a finite dimensional vector space, is isomorphic to $\mathbb{K}^n$ by the choice of a basis $\{v_i\}_{i=1,\ldots,n}$. Consider the identity map $I : F \to F$. The basis defines continuous functionals $\lambda_i : F \to \mathbb{K}$ by $\lambda_i(\sum_{i=1}^n c_i v_i) = c_i$ for $i = 1, \ldots, n$. (The continuity is for example obvious with the supremum norm but does not depend on the choice of norm in the finite dimensional setting.) Now extend each λ_i by the Hahn-Banach theorem to continuous functionals $\Lambda_i : E \to \mathbb{K}$ and consider the continuous linear map $P : E \to F$ defined by $P(v) = \sum_{i=1}^n \Lambda_i(v) v_i$. *Claim:* $G := \ker(P) = \{v \in E : P(v) = 0\}$ is a closed subspace of E such that $E = F + G$ and $F \cap G = \{0\}$. *Proof.* $\ker(P)$ is obviously a subspace, which is closed because $\ker(P) = P^{-1}(\{0\})$ and $\{0\} \subset F$ is a closed subset. For each $v \in E$, $v - P(v) \in G$ because $P(v - P(v)) = P(v) - P(P(v)) = 0$ because $P(v) \in F$ and $P|F = I$ and thus $P(P(v)) = P(v)$. Thus $v - P9v) = w \in G$ and $v = P(v) + w$. This shows $E = F + G$. Moreover, if $v \in F \cap G$ then $P(v) = v$ and $P(v) = 0$ and thus $v = 0$. This shows $F \cap G = \{0\}$.