Solutions Homework Assignment 5, MATH 515, Spring 09

Problem 11) For $p = 1$ this is follows by induction from the usual triangle inequality and nothing has to be shown. For $p > 1$ let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. We can assume that not all a_k and not all b_k are 0, and also not $a_k + b_k = 0$ for all k. It follows from the triangle inequaltiy that

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \sum_{k=1}^{n} |a_k + b_k|^{p-1}|a_k| + \sum_{k=1}^{n} |a_k + b_k|^{p-1}|b_k|.$$

Now apply Hölder's inequality from Problem 10 to each of the two sums in the form for real nonnegative numbers c_k, d_k, $k = 1, \ldots, n$:

$$\sum_{k=1}^{n} c_k d_k \leq \left(\sum_{k=1}^{n} {c_k}^q\right)^{1/q} \left(\sum_{k=1}^{n} {d_k}^p\right)^{1/p},$$

so we actually use the real version, with $c_k = |a_k + b_k|^{p-1}$, and $d_k = |a_k|$ in the first sum respectively $d_k = |b_k|$ in the second sum:

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \left(\sum_{k=1}^{n} |a_k + b_k|^{(p-1)q}\right)^{1/q} \left(\left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}\right)$$

and using $(p-1)q = p$ it follows

$$= \left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/q} \left(\left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}\right)$$

Now divide both sides by $\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/q}$ with $1 - \frac{1}{q} = \frac{1}{p}$ to get the inequality. If (i),(ii) or (iii) holds then we have equality. This is obvious for (i). For (ii) not that $\left(\sum_{k=1}^{n} (1+t)^p |a_k|\right)^{1/p} = \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |t|^p |a_k|^p\right)^{1/p}$ and because $t \geq 0$ we get $(|t|^p)^{1/p} = t$. Conversely, for $p > 1$ by Problem 10, equality in the Hölder inequality implies $d_k = 0$ for all k, or ${c_k}^q = s{d_k}^p$ for some s. Now, with the definitions above we get all $a_k = 0$ or $|a_k + b_k| = s|a_k| = |a_k + ta_k|$ for some real t. But then equality can only hold if $t \geq 0$ and $b_k = ta_k$ for all k. If $p = 1$ this follows because equality in the triangle inequality $|a_k + b_k| = |a_k| + |b_k|$ can only hold if b_k is a real nonnegative multiple of a_k.

Problem 12) (a) Now $|.|_p$ for $1 \leq p < \infty$ is a norm on $\mathbb{C}^n$. The only nontrivial part, namely the triangle inequality is just the Minkowski inequality for $p > 1$, while for $p = 1$ this is the supremum norm for maps $\{1, \ldots, n\} \to \mathbb{C}$. Because $|.|_p$ is also a norm if we consider $\mathbb{C}^n$ as a real vector space $\mathbb{R}^{2n}$ and all norms on finite dimensional real vector spaces are equivalent, completeness is immediate from completeness with respect to Euclidean norms.

(b) The fact that ℓ_p is a normed vector space for $1 \leq p < \infty$ is clear for $p = \infty$ because this is the supremum norm for bounded maps $\mathbb{N} \to \mathbb{C}$. For

$p > 1$ the triangle inequality also follows from Minkowski's inequality by letting $n \to \infty$ and noticing that by assumption all resulting series converge. The case $p = 1$ is Problem 2, where also completeness has been shown. Now let $\{x_k\}$ be a Cauchy sequence in ℓ_p, i. e. for each $\varepsilon > 0$ there exists N such that $|x_n - x_m|_p < \varepsilon$ for $n, m > N$. If we let $x_k = \{x_{ki}\}_i$ with $x_{ki} \in \mathbb{C}$ for all $k, i \in \mathbb{N}$ then $|x_n - x_m|_p = \left(\sum_{i=1}^{\infty} |x_{ni} - x_{mi}|^p\right)^{1/p}$ for $1 \leq p < \infty$, respectively $|x_n - x_m|_\infty = \sup_i |x_{ni} - x_{mi}|$. In each case it follows that for fixed i the sequence $\{x_{ki}\}_k$ is a Cauchy sequence in $\mathbb{C}$ and thus converges to some element denoted $y_i \in \mathbb{C}$. If $y = \{y_i\}$ then we want to show $y \in \ell_p$ and $x_n \to y$ in ℓ_p. Suppose $1 \leq p < \infty$. In

$$\left(\sum_{i=1}^{N} |x_{ni} - x_{mi}|^p\right)^{1/p} \leq |x_n - x_m|_p < \varepsilon$$

by letting $m \to \infty$ and then $N \to \infty$ we get $|x_n - y| \leq \varepsilon$ and thus $x_n \to y$ in ℓ_p. Similarly for $p = \infty$ in $\sup_i |x_{ni} - x_{mi}| < \varepsilon$ by letting $m \to \infty$ we get $|x_n - y| \leq \varepsilon$. Then because $x_n \in \ell_p$ and $x_n - y \in \ell_p$ we get also $(x_n - (x_n - y) = y \in \ell_p$, and $x_n \to y$ in ℓ_p.

Problem 13) The elements of E/F are the *cosets* $\overline{x} := x + F = \{x + y : y \in F\} \subset E$. Then $\overline{x} + \overline{y} := \overline{x + y}$ and $c\overline{x} := \overline{cx}$ define the vector space structure on E/F. Note that the elements $\overline{x}$ are equivalence classes of the equivalence relation on E defined by $x \sim y$ if $x - y \in F$ (this is an equivalence relation: $x \sim x$ because $x - x = 0 \in F$, if $x \sim y$ then $x - y \in F$ so also $-(x - y) = y - x \in F$ and thus $y \sim x$, if $x \sim y$ and $y \sim z$ then $x - y \in F$ and $y - z \in F$ thus also $x - y + (y - z) = x - z \in F$ and thus $x \sim z$.) The 0-vector in E/F is the equivalence class $\overline{0} = \overline{y}$ for all $y \in F$. *Claim:* $|\overline{x}| := \inf_{y \in F} |x + y|$ defines a norm on E/F such that the map $E \ni x \mapsto \overline{x} = x + F$ is linear and continuous. *Proof.* NVS1: $\overline{x} = 0 \implies x \in F \implies \inf_{y \in F} |x + y| = 0$ because we can take $y = -x \in F$ and then $x + y = 0$. Conversely, $\inf_{y \in F} |x + y| = 0$ implies that there exists a sequence $\{y_n\}$ in F such that $|x + y_n| \to 0$. Now $\{y_n\}$ is a Cauchy sequence in E because $|y_n - y_m| = |y_n + x - (y_m + x)| \leq |y_n + x| + |y_m + x| < \varepsilon$ for $n, m > N$. Thus y_n converges to some $y \in E$, which is in F because F is closed in E. (This actually reproves in the special case that for $A, B \subset \mathbb{R}$, $\inf(A + B) = \inf(A) + \inf(B)$, which can be applied directly and simplifies the proof.) Thus there exists a $y \in F$ such that $|x + y| = 0$ and so $x = -y \in F$, which implies $x \in F$, and thus $\overline{x} = 0 \in E/F$. NVS2: $|c\overline{x}| = |\overline{cx}| = \inf_{y \in F} |cx + y| = \inf_{z \in F} |cx + cz|$, because F is a subspace and thus $y \in F \iff cy \in F$ for $c \neq 0$, and thus $\inf_{y \in F} |cx + cy| = |c| \inf_{y \in F} |x + y| = |c|\ |\overline{x}|$. NVS3: Now for $x, z \in E$ and $y \in F$, by the definition of the infimum, we know that $|\overline{x}| = \lim |x + y_n|$ and $|\overline{z}| = \lim |z + y'_n|$ for two sequences $\{y_n\}, \{y'_n\}$ in F. Thus for each $\varepsilon > 0$ we get $\inf_{y \in F} |x + z + y| \leq |x + z + y_n + y'_n| \leq |x + y'_n| + |z + y'_n| \leq |\overline{x}| + |\overline{z}| + 2\varepsilon$. Because this holds for each $\varepsilon > 0$ we get $|\overline{x} + \overline{z}| \leq |\overline{x}| + |\overline{z}|$. Linearity of $x \mapsto \overline{x}$ is by the very definition. Continuity follows because $|\overline{x}| = \inf_{y \in F} |x + y| \leq |x|$, so in fact the natural map is bounded by 1.