Solutions Homework Assignment 2, MATH 515, Spring 09

Problem 4) A semi-norm σ always satisfies $\sigma(0) = \sigma(0 \cdot 0) = |0|\sigma(0) = 0$. Thus the difference between a norm and a seminorm only is in $\sigma(x) = 0 \Longrightarrow x = 0$.
(a) Note that $(\sigma_1+\sigma_2)(x) := \sigma_1(x)+\sigma_2(x)$. Thus $(\sigma_1+\sigma_2)(x) \geq 0$ for all $x \in E$. Also $(\sigma_1 + \sigma_2)(cx) = \sigma_1(cx) + \sigma_2(cx) = |c|\sigma_1(x) + |c|\sigma_2(x) = |c|(\sigma_1 + \sigma_2)(x)$, and $(\sigma_1 + \sigma_2)(x + y) = \sigma_1(x + y) + \sigma_2(x + y) \leq (\sigma_1(x) + \sigma_1(y)) + (\sigma_2(x) + \sigma_2(y)) = (\sigma_1 + \sigma_2)(x) + (\sigma_1 + \sigma_2)(y)$. Similarly for $\lambda \geq 0$a real number, and σ a seminorm, it follows that $\lambda\sigma$ is a seminorm (the general result follows then immediately). $\lambda\sigma(x) \geq 0$ because $\lambda \geq 0$ and $\sigma(x) \geq 0$ for all x. Also for a scalar c we get $(\lambda\sigma)(cx) = \lambda\sigma(cx) = \lambda|c|\sigma(x) = |c|\lambda\sigma(x) = |c|(\lambda\sigma)(x)$. Finally $(\lambda\sigma)(x + y) = \lambda\sigma(x + y) \leq \lambda\sigma(x) + \lambda\sigma(y)$. Note that $\lambda \geq 0$ is essential also for the proof of the triangle inequality. The indudction proof now goes as follows: For $n = 2$ we have already proved the result. Assume it is proved for some n, and consider $\lambda_1\sigma_1 + \ldots + \lambda_n\sigma_n$. Then $\lambda_1\sigma_1 + \ldots + \lambda_n\sigma_n$ is a seminorm by induction assumption, and $\lambda_{n+1}\sigma_{n+1}$ is a seminorm as shown above. Then the sum is a seminorm as shown above.

(b) Certainly $\max(\sigma_1, \sigma_2)(x) \geq 0$ because $\sigma_1(x) \geq 0$ and $\sigma_2(x) \geq 0$. Also $\max(\sigma_1, \sigma_2)(cx) = \max(\sigma_1(cx), \sigma_2(cx)) = \max(|c|\sigma_1(x), |c|\sigma_2(x)) = |c|max(\sigma_1(x), \sigma_2(x))$ because the seminorms and $|c|$ are nonnegative. Now $\sigma_i(x + y) \leq \sigma_i(x) + \sigma_i(y) \leq \max(\sigma_1(x), \sigma_2(x)) + \max(\sigma_1(y), \sigma_2(y))$ for $i = 1, 2$, and now the triangle inequality for $\max(\sigma_1, \sigma_2)$ follows by taking the maximum for $i = 1, 2$ in the last inequality.

(c) Because of (a) and (b) we only have to show that for $\sigma = \sigma_1 + \sigma_2$ or $\max(\sigma_1, \sigma_2)$, $\sigma(x) = 0$ implies $x = 0$. But $\sigma_1(x) + \sigma_2(x) = 0$ implies $\sigma_1(x) = 0$ because $\sigma_i(x) \geq 0$ for $i = 1, 2$. Since σ_1 is a norm we conclude $x = 0$. Similarly $0 = \max(\sigma_1(x), \sigma_2(x)) \geq \sigma_1(x) \geq 0$ gives $\sigma_1(x) = 0$ and $x = 0$ because σ_1 is a norm.

(d) Let $F := \{x \in E : \sigma(x) = 0\}$. Because $\sigma(0) = 0$ we have $0 \in F$. Thus $F \neq \emptyset$. Moreover, for a scalar c and $x \in F$, we have $\sigma(cx) = |c|\sigma(x) = 0$ thus $cx \in F$. Also if $x, y \in F$ then $\sigma(x) = 0$ and $\sigma(y) =)$, and from $0 \leq \sigma(x + y) \leq \sigma(x) + \sigma(y) = 0$ it follows that $\sigma(x + y) = 0$ and thus $x + y \in F$.

Problem 5) (a) $B(X, E)$ is equipped with the supremum norm defined by $||f|| = \sup_{x\in X} |f(x)|$, where $|.|$ is the norm on E. We have to show that if $f_n \to f$ uniformly, i. e. in the supremum norm with $f_n \in BC(X, E)$, $f \in B(X, E)$ then $f \in BC(X, E)$. (Note that $B(X, E)$ is a metric space, and a subset S of a metric space X is closed if the limit of each sequence in the subset is already in the subset. In fact if $x \in \partial S$ then there exists a sequence in S, converging in X, with limit x. If $x \in S$ take the constant sequence. Otherwise choose elements in S within balls of radius $\frac{1}{n}$ for $n = 1, 2, \ldots$ to construct the sequence.) We will need the following

Lemma. *A map $f : X \to E$ with X a topological space and E a normed vector* *space is continuous if and only if for each $x \in E$ and each $\varepsilon > 0$ there exists an open neighborhood U of x such that for all $y \in U$: $|f(x) - f(y)| < \varepsilon$.*

Proof. $\Longrightarrow$: The ball $B_\varepsilon(f(x)) = \{w \in E : |w - f(x)| < \varepsilon\}$ is open and thus its inverse image under f is open and contains x. For each $y \in f^{-1}(B_\varepsilon(f(x))) =: U$ we have $f(y) \in B_\varepsilon(f(x))$ and thus $|f(y) - f(x)| < \varepsilon$. $\Longleftarrow$: Let $V \subset E$ be open and $x \in f^{-1}(V)$. There exists $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subset V$. For this ε find U_x, a neighborhood of x, such that $f(U_x) \subset B_\varepsilon(f(x)) \subset V$. Then $U_x \subset f^{-1}(V)$. In this way $f^{-1}(V) = \cup_{x\in f^{-1}(V)} U_x$ is seen to be open in X. ■

Now assume $x \in X$, $\varepsilon > 0$ and $f_n \to f$ uniformly. So there exists N such that for all $n > N$ we have $|f_n(u) - f(u)| < \varepsilon/3$ for all $u \in X$. Pick such an n. Because f_n is continuous we can find a neighborhood U of x such that $|f_n(x) - f_n(y)| < \varepsilon/3$ for all $y \in U$. Then for all $y \in U$ we have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon.$$

Thus $f \in BC(X, E)$ follows from the lemma above.

(b) Suppose that $\{f_n\}$ is a Cauchy sequence in $B(X, E)$. Then for each $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence in E, which converges to some element $f(x) \in E$. We have to show that $f_n \to f$ uniformly and $f \in B(X, E)$ Now for given $\varepsilon > 0$ and all $x \in X$ we know $|f_n(x) - f_m(x)| < \varepsilon$. Let $m \to \infty$ in this inequality to get for all $x \in X$: $|f_n(x) - f(x)| \leq \varepsilon$. This shows $f_n \to f$ uniformly and $f_n - f \in B(X, E)$. Since $f_n \in B(X, E)$ it follows that also $f_n - (f_n - f) = f \in B(X, E)$ because $B(X, E)$ is a vector space.

(c) If X is compact then consider for each $f \in C(X, E)$ the map $X \ni x \mapsto g(x) := |f(x)| \in \mathbb{R}$. Because the composition of continuous functions is continuous and the norm function $E \to \mathbb{R}$ is continuous, $f(X) \subset \mathbb{R}$ is compact and thus closed and bounded. Thus $f \in BC(X, E)$ by definition. (Note that (i) for each normed vector space E, $|.| : E \to \mathbb{R}$ is continuous because $||x| - |y|| \leq |x - y|$ and thus for given $\varepsilon > 0$ we can choose $\delta = \varepsilon$ and conclude from $|x - y| < \delta$ that $||x| - |y|| < \varepsilon$; (ii) A function f from a set S into a normed vector space E is bounded $\Longleftrightarrow |f(x)| \leq M$ for all $x \in E \Longleftrightarrow \{|f(x)| : x \in S\} \subset \mathbb{R}$ is bounded $\Longleftrightarrow |.| \circ f : S \to \mathbb{R}$ is bounded.) **Alternatively:** $f(X) \subset E$ is compact and thus totally bounded. It follows that $f(E)$ is bounded and thus f is bounded by definition. (The first proof is more elementary because the implication compact $\Longrightarrow$ totally bounded in metric spaces is a rather nontrivial fact.)