Solutions Homework Assignment 1, MATH 515, Spring 09

Problem 1) Let S be a subset of a topological space X and let $\overline{S} = S \cup \partial S$ denote the closure of S. Recall $x \in \partial S \iff \forall$ open neighborhoods U of x: $U \cap S \neq \emptyset$ and $U \cap (X \setminus S) \neq \emptyset$.

(a) $\overline{S}$ is closed. *Proof.* Let $x \in X \setminus \overline{S}$. Since $x \notin \partial S$ there exists an open neighborhood U of x with $U \cap S = \emptyset$. If $U \cap \partial S \neq \emptyset$ then U would be also be open neighborhood of $y \in U \cap \partial S$ and so $U \cap S \neq \emptyset$, which is a contradiction. Thus $U \subset X \setminus \overline{S}$, and $\overline{S}$ is closed. +

(b) If S, T are subsets of X and $S \subset T$, show that $\overline{S} \subset \overline{T}$. *Proof.* $S \subset T \subset \overline{T}$. Let $x \in \partial S$, $x \notin S$ and let N be open neighborhood of x. If $x \in T$ then $x \in \overline{T}$ and we are done. But N contains a point of S, which is also in T. So N contains points in T and $X \setminus T$. Thus $S \cup \partial S \subset \overline{T}$.

(c) If S, T are subsets of X, show that $\overline{S \cup T} = \overline{S} \cup \overline{T}$. *Proof.* From $S \subset S \cup T$ and $T \subset S \cup T$ we get, using (b), $\overline{S} \subset \overline{S \cup T}$ and $\overline{T} \subset \overline{S \cup T}$, and thus $\overline{S} \cup \overline{T} \subset \overline{S \cup T}$. Of course $S \cup T \subset \overline{S} \cup \overline{T}$. If $x \in \partial(S \cup T)$ and N is a neighborhood of x then N contains points in $S \cup T$ and in $X \setminus (S \cup T) = (X \setminus S) \cup (X \setminus T)$. So we find some point in S and some point not in S, or we find some point in T and some point not in T. In any case $\partial(S \cup T) \subset \partial S \cup \partial T \subset \overline{S} \cup \overline{T}$.

(d) Show that $\overline{\overline{S}} = \overline{S}$. *Proof.* $S \subset \overline{S} \Longrightarrow \overline{S} \subset \overline{\overline{S}}$ by (b) above. To show $\overline{\overline{S}} \subset \overline{S}$ it suffices to show $\partial \overline{S} \subset \partial S$ because $\partial S \subset \overline{S}$. If N is open neighborhood of $x \in \partial \overline{S}$ then N contains a point in $X \setminus \overline{S}$ which is of course in $X \setminus S$, and a point in $\overline{S}$. If this second point is in ∂S and not in S then there exists another point of S in N by the definition of ∂S. In each case $x \in \partial S$.

(f) If $S \subset T \subset \overline{S}$, prove that $\overline{T} = \overline{S}$. *Proof.* $S \subset T \Longrightarrow \overline{S} \subset \overline{T}$ by (b) above. Also $T \subset \overline{S} \Longrightarrow \overline{T} \subset \overline{\overline{S}} = \overline{S}$ by (b) and (d).

Problem 2) Let l^1 be the set of sequences $\alpha = \{\alpha_n\}$ of real numbers such that $\sum |a_n|$ converges, and define $|\alpha| := \sum |\alpha_n|$.

(a) *Claim.* l^1 with this norm is a complete normed vector space. *Proof.* Recall that for α, β sequences in $\mathbb{R}$ and $\lambda \in \mathbb{R}$, $\alpha + \beta := \{(\alpha_n + \beta_n)\}$, $\lambda\alpha = \{(\lambda\alpha_n)\}$ and the constant zero sequence is the 0 in the vector space of all sequences in $\mathbb{R}$. Note that $0 \in l^1$. The inequality:

$$|\alpha + \beta| = \sum |\alpha_n + \beta_n| \leq \sum |\alpha_n| + \sum |\beta_n| = |\alpha| + |\beta|$$

shows both that l^1 is closed with respect to addition and NVS3 holds for $|.|$. Also

$$|\lambda\alpha| = \sum |\lambda\alpha_n| = \sum |\lambda| \, |\alpha_n| = |\lambda| \sum |\alpha_n|$$

shows that l^1 is closed with respect to multiplication by scalars and NVS2 holds. NVS1 is true because $\sum |\alpha_n| \geq 0$ and if $\sum |\alpha_n| = 0$ then $\alpha_n = 0$ for all n and

α is the zero sequence. Let $x_k := \{\alpha_n^{(k)}\}$ be a Cauchy sequence in l^1. Then for each positive integer n and given $\varepsilon > 0$ there exists K such that for all $k, \ell > K$

$$|\alpha_n^{(k)} - \alpha_n^{(\ell)}| \leq |x_k - x_\ell| < \varepsilon.$$

Thus for each positive integer n, the sequence $\{\alpha_n^{(k)}\}_k$ is a Cauchy sequence of real numbers and has a limit α_n in $\mathbb{R}$. Let $\alpha = \{\alpha_n\}$ be the limit sequence. Now for each positive integer M we have for $k, \ell > K$

$$\sum_{n=1}^{M} |\alpha_n^{(k)} - \alpha_n^{(\ell)}| \leq |x_k - x_\ell| < \varepsilon$$

Now let $\ell \to \infty$ to get $\sum_{n=1}^{M} |\alpha_n^{(k)} - \alpha_n| \leq \varepsilon$. This holds for all M so can let $M \to \infty$ to get $|x_k - \alpha| \leq \varepsilon$ for all $k > K$. Thus $\{x_k\}$ converges to α in the norm on l^1. Note that $x_k - \alpha \in l^1$ implies $x_k - (x_k - \alpha) = \alpha \in l^1$.

(b) Let $\beta = \{\beta_n\}$ be a fixed sequence and let K be the set of all $\alpha \in l^1$ such that $|\alpha_n| \leq |\beta_n|$ for all n. *Claim.* K is compact. *Proof.* Let x_k be a sequence in K and $\varepsilon > 0$. Then for each n the sequence $\{\alpha_n^{(k)}\}_k$ is a sequence of real number in $[-b_n, b_n]$, which is compact, and thus has a convergent subsequence. Construct a subsequence of x_k by induction over n. First choose a subsequence of x_k such that the first component converges to some $\alpha_1 \in [-\beta_1, \beta_1]$. Then choose a subsequence of this subsequence such that the first two components of x_k converge to $\alpha_1 \in [-\beta_1, \beta_1]$ respectively $\alpha_2 \in [-\beta_2, \beta_2]$. This defines a sequence $\alpha = \{\alpha_n\}$. Now for $\varepsilon > 0$ we find N such that $\sum_{n>N} |\beta_n| < \varepsilon$, and thus $\sum_{n>N} |\alpha_n^{(k)} - \alpha_n| < \varepsilon$ (note that for each n, $\alpha_n^{(k)}, \alpha \in [-\beta_n, \beta_n]$). Now for $n = 1, \ldots, N$ we know that the components of the subsequence constructed above converge to the corresponding components of α. Thus for each $n = 1, \ldots N$ find $K_1, \ldots K_n$ such that $|\alpha_n^{(k_i)} - \alpha_n| < \frac{\varepsilon}{N}$ for $i > K_n$ and $x_{k_i} = \{\alpha_n^{(k_i)}\}_i$ is the subsequence constructed above. Then for $i > \max_n(K_n, N)$ it follows that $|x_{k_i} - \alpha| \leq 2\varepsilon$. Thus we have constructed a subsequence of $\{x_k\}$ converging to some point in K. Thus K is sequentially compact and thus compact (because l^1 is a metric space).

Let S_1 be the unit sphere in l^1. Then define a sequence $x_k = \{\alpha_n^{(k)}\} \in l^1$ by $\alpha_n^{(k)} = 1$ if $k = n$ and $= 0$ otherwise. Then for $k \neq \ell$, $|x_k - x_\ell| = 2$ and thus $\{x_k\}$ cannot have a convergent subsequence.

Problem 3) (a) Let (X, d) be a metric space and $d'(x, y) := \min(d(x, y), 1)$. *Claim.* (X, d') is a metric space and convergence with respect to d and convergence with respect to d' is the same. *Proof.* DIS1: $d'(x, y) \geq 0$ is clear. If $d'(x, y) = 0$ then $d"(x, y) = d(x, y)$ thus $x = y$. DIS2: Because $d(x, y) = d(y, x)$ also $d'(x, y) = d'(y, x)$. DIS3: If one of $d(x, z)$ or $d(y, z)$ is > 1 then $d'(x, z) = 1$ or $d'(y, z) = 1$ and thus $d'(x, y) \leq 1 \leq d(x, z) + d(y, z)$. If both $d(x, z)$ and $d(y, z)$ are ≤ 1 then $d'(x, y) \leq d(x, y) \leq d(x, z) + d(y, z) = d'(x, z) + d'(y, z)$ holds by DIS3 for d. Note that in the definition of convergence we can always assume $\varepsilon < 1$ (if it works for all $\varepsilon < 1$ it will work for all ε) and then $d' = d$.

(b) Note that a sequence is Cauchy with respect to d if and only if it is Cauchy with respect to d' (same argument as above). So if (X, d) is complete then any Cauchy sequence with respect to d converges with respect to d. Consider a Cauchy sequence with respect to d'. This is Cauchy with respect to d so converges with respect to d and thus converges with respect to d'. The same argument works with d, d' interchanged.

(c) For each $x \in X$ define $f_x : X \to \mathbb{R}$ by $f_x(y) := d(x, y)$. Let $||.||$ be the supremum norm. *Claim.* $d(x, y) = ||f_x - f_y||$. *Proof.* $|d(x, z) - d(z, y)| \leq d(x, y)$ follows from DIS3 for each metric. Thus $||f_x - f_y|| = sup_{z \in X} |d(x, z) - d(z, y)| \leq d(x, y)$. Conversely, $sup_{x \in X} |d(x, z) - d(y, z)| \geq |d(x, y) - d(y, y)| = d(x, y)$.

for $a \in X$ fixed let $g_x := f_x - f_a$. *Claim.* $x \mapsto g_x$ is a distance-preserving embedding of X into the normed vector space of bounded functions on X. *Proof.* $||g_x - g_y|| = ||f_x - f_a - (f_y - f_a)|| = ||f_x - f_y|| = d(x, y)$ so $x \mapsto g_x$ is distance-preserving. From the implications $g_x = g_y \implies f_x - f_a = f_y - f_a \implies f_x = f_y \implies \forall z \; d(x, z) = d(y, z) \implies 0 = d(x, x) = d(y, x) \implies x = y$ it follows that $x \mapsto g_x$ is one-to-one. Note that for each x, g_x is a bounded function on X, because $|g_x(z)| = |f_x(z) - f_a(z)| = |d(x, z) - d(a, z)| \leq d(x, a)$.