Takehome Exam, MATH 515, Spring 09

**Problem 1** (75 pts) For \( a < b \) let \( W[a, b] \) be the vector space of continuously differentiable functions on \([a, b]\) with values in \( \mathbb{C} \). For \( f, g \in W[a, b] \) let

\[
(f, g)_W := \int_a^b (f(t)\overline{g(t)}) + f'(t)g'(t))dt
\]

and \( |f|_W := \sqrt{\langle f, f \rangle_W} \).

(a) Show that \( \langle \cdot, \cdot \rangle_W \) defines a positive definite hermitian form on \( W[a, b] \), and thus \( |.|_W \) is a norm on \( W[a, b] \).

(b) Consider \( W[0, 1] \). Let \( \chi_n \in W[0, 1] \) be defined by \( \chi_n(t) := e^{2\pi i nt}, n \in \mathbb{Z} \) and \( 0 \leq t \leq 1 \). Prove that \( \langle \chi_n, \chi_m \rangle_W = 0 \) when \( n \neq m \), and \( |\chi_n - \chi_m|_W = \sqrt{2 + 4\pi^2(n^2 + m^2)} \).

(c) Prove that in \( W[0, 1] \)

\[
(f, \cosh)_W = f(1) \sinh(1)
\]

and deduce that

\[
\{ f \in W[0, 1] : f(1) = 0 \}
\]

is a closed subspace of \( W[0, 1] \).

(d) Prove that \( W[a, b] \) is not a Hilbert space. Hint: Consider indefinite integrals of the sequence of functions \( f_n(t) = 0 \) for \( t \leq \frac{1}{2} - \frac{1}{n} \), \( f_n(t) = 1 \) for \( t \geq \frac{1}{2} \) and the graph of \( f_n(t) \) is the line joining \((\frac{1}{2} - \frac{1}{n}, 0)\) and \((\frac{1}{2}, 1)\) for \( t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \).

**Problem 3** (75 pts) page 108, Chapter V, §6, Exercise 8. The formula in (b) should be:

\[
\langle x, y \rangle = \frac{1}{2}(|x + y|^2 - |x|^2 - |y|^2) + i\frac{1}{2}(|x + iy|^2 - |x|^2 - |y|^2)
\]

**Problem 3** (Bonus Problem 50 pts) page 108, Chapter V, §6, Exercise 7