**Takehome Exam, MATH 515, Spring 09**

**Problem 1) ( 75 pts)** For $a < b$ let $W[a,b]$ be the vector space of continuously differentiable functions on $[a,b]$ with values in $\mathbb{C}$. For $f,g \in W[a,b]$ let

$$\langle f,g\rangle_W := \int_a^b (f(t)\overline{g(t)} + f'(t)\overline{g'(t)})dt$$

and $|f|_W := \sqrt{\langle f,f\rangle_W}$.

(a) Show that $\langle\ ,\ \rangle_W$ defines a positive definite hermitian form on $W[a,b]$, and thus $|.|_W$ is a norm on $W[a,b]$.

(b) Consider $W[0,1]$. Let $\chi_n \in W[0,1]$ be defined by $\chi_n(t) := e^{2\pi int}$, $n \in \mathbb{Z}$ and $0 \le t \le 1$. Prove that $\langle \chi_n, \chi_m\rangle_W = 0$ when $n \neq m$, and $|\chi_n - \chi_m|_W = \sqrt{2 + 4\pi^2(n^2+m^2)}$.

(c) Prove that in $W[0,1]$

$$\langle f, \cosh\rangle_W = f(1)\sinh(1)$$

and deduce that

$$\{f \in W[0,1] : f(1) = 0\}$$

is a closed subspace of $W[0,1]$.

(d) Prove that $W[a,b]$ is not a Hilbert space. Hint: Consider indefinite integrals of the sequence of functions $f_n(t) = 0$ for $t \le \frac{1}{2} - \frac{1}{n}$, $f_n(t) = 1$ for $t \ge \frac{1}{2}$ and the graph of $f_n(t)$ is the line joining $(\frac{1}{2} - \frac{1}{n}, 0)$ and $(\frac{1}{2}, 1)$ for $t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$.

**Problem 3) (75 pts)** page 108, Chapter V, §6, Exercise 8. The formula in (b) should be:

$$\langle x,y\rangle = \frac{1}{2}(|x+y|^2 - |x|^2 - |y|^2) + i\frac{1}{2}(|x+iy|^2 - |x|^2 - |y|^2)$$

**Problem 3) (Bonus Problem 50 pts)** page 108, Chapter V, §6, Exercise 7