Homework Assignment 5, MATH 515, Spring 09

Problem 11) (8 pts) Show the following *Minkowski inequality*:
For $1 \leq p < \infty$, $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$:

$$(\sum_{k=1}^{n} |a_k + b_k|^p)^{1/p} \leq (\sum_{k=1}^{n} |a_k|^p)^{1/p} + (\sum_{k=1}^{n} |b_k|^p)^{1/p}$$

with equality if and only if one of the following holds: (i) all a_k are 0, (ii) $b_k = ta_k$ for all a_k and $t \geq 0$, (iii) $p = 1$ and for each k, $a_k = 0$ or $a_k = t_k b_k$ for some $t_k \geq 0$.

Hint: For $p > 1$ consider $|a_k + b_k|^p \leq |a_k + b_k|^{p-1}(|a_k| + |b_k|)$ and two applications of Hölder's inequality with q such that $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 12) (8 pts) (a) For $x = (x_1, \ldots x_n) \in \mathbb{C}^n$ put $|x|_p := (\sum_{k=1}^{n} |x_k|^p)^{1/p}$ for $1 \leq p < \infty$, and $|x|_\infty := \max_{1 \leq i \leq n} |x_i|$. Show that $\mathbb{C}^n$ with $|.|_p$ is a Banach space.

(b) For $1 \leq p \leq \infty$ let ℓ_p denote the set of sequences $x = \{x_n\}$ with $x_n \in \mathbb{C}$ for $n \in \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} |x_n|^p$ converges if $p < \infty$, respectively $\{x_n\}$ is a bounded sequence if $p = \infty$. Show that the sets ℓ_p with the norm $|x|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$ if $p < \infty$, respectively $|x|_\infty := \sup_{n \in \mathbb{N}} |x_n|$, are Banach spaces.

Problem 13) (8 pts) page 91, Chapter IV, §6, Exercise 4