

### Homework Assignment 4, MATH 515, Spring 09

**Problem 8) (4 pts)** A subset  $D$  of a real vector space  $E$  is *convex* if for all  $x, y \in D$  and  $0 < t < 1$  it follows  $tx + (1-t)y \in D$ . Show the following: If  $x_1, \dots, x_n \in D$  then also  $t_1x_1 + \dots + t_nx_n \in D$  for all  $t_1, \dots, t_n > 0$  such that  $\sum_{i=1}^n t_i = 1$ . Conversely, for given  $x_1, \dots, x_n \in E$  the set of all combinations as above is a convex set.

**Problem 9) (12 pts)** (a) Let  $D \subset \mathbb{R}^n$  be convex. A function  $f : D \rightarrow \mathbb{R}$  is called *convex* if  $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$  for all  $x, y \in D$  and  $0 < t < 1$ . The function  $f$  is called *strictly convex* if strict inequality holds whenever  $x \neq y$  and  $0 < t < 1$ . The function  $f$  is *concave* respectively *strictly concave* if  $-f$  is convex respective *strictly convex*. Show the following: If  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable then  $f$  is convex  $\iff f'$  is increasing, and  $f$  is concave  $\iff f'$  is decreasing. In particular, if  $f$  is twice differentiable and  $f'' \geq 0$  then  $f$  is convex, while if  $f'' \leq 0$  then  $f$  is concave. Also  $f'' > 0$  then  $f$  is strictly convex and if  $f'' < 0$  then  $f$  is strictly concave.

(b) Prove *Jensen's inequality*: For  $D$  a convex subset of a vector space  $E$  and  $f : D \rightarrow \mathbb{R}$  a concave function show that

$$\sum_{i=1}^n t_i f(x_i) \leq f\left(\sum_{i=1}^n t_i x_i\right)$$

whenever  $x_1, \dots, x_n \in D$ ,  $t_1, \dots, t_n \in (0, 1)$  and  $\sum_{i=1}^n t_i = 1$ . Furthermore, if  $f$  is strictly concave then equality holds if and only if  $x_1 = \dots = x_n$ . Hint: induction on  $n$ .

(c) Apply (b) to  $f(x) = \log(x)$  to show that if  $a_1, \dots, a_n \geq 0$  and  $p_1, \dots, p_n > 0$  are real numbers with  $\sum_{i=1}^n p_i = 1$  then

$$\prod_{i=1}^n a_i^{p_i} \leq \sum_{i=1}^n p_i a_i$$

with equality if and only if  $a_1 = \dots = a_n$ .

(d) Prove that for  $a, b \geq 0$  and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  it follows  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$  with equality if and only if  $a^p = b^q$ .

**Problem 10 (8 pts)** Prove *Hölder's inequality*: Suppose  $p, q > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Then for complex numbers  $a_1, \dots, a_n, b_1, \dots, b_n$  we have

$$\left| \sum_{k=1}^n a_k b_k \right| \leq \left( \sum_{k=1}^n |a_k|^p \right)^{1/p} \left( \sum_{k=1}^n |b_k|^q \right)^{1/q}$$

with equality if and only if all  $a_k$  are 0 or  $|b_k|^q = t|a_k|^p$  and  $a_k b_k = e^{i\theta} |a_k b_k|$  for all  $k$  and some  $t$  and  $\theta$ . Hints: Reduce to the case  $\sum_{k=1}^n |a_k|^p = \sum_{k=1}^n |b_k|^q = 1$ .