

DEFORMATION OF HOMOTOPY INTO ISOTOPY IN ORIENTED 3-MANIFOLDS

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ABSTRACT. We will show that deformation quantization in skein theory of oriented 3-manifolds is induced from a *topological* deformation quantization of the fundamental 2-groupoid of the space of immersions of circles in M . The structure of skein module and its relations with string topology homomorphisms appear through representations of the groupoid structure into the set of objects. The deformation of the fundamental 2-groupoid is defined by the singularity stratification, the quantization by passage to isotopy classes. Several explicit properties and computations of skein modules are proved. It will be shown that local systems on the space of immersions are important for the understanding of HOMFLY oriented and framed skein theory. The passage from Conway to Jones skein theory is described on the categorical level.

1. INTRODUCTION AND SUMMARY OF RESULTS

In this paper we will develop categorical tools to study the stratified topology of mapping spaces. It will be shown how the structures of skein modules and string topology naturally are induced from the categorical level. This gives both a theoretical understanding and new tools in the computation of skein modules of links in oriented 3-manifolds.

Recall that a skein module of an oriented 3-manifold M is the quotient of a free module with basis a set of links (possibly including singular links) in M by a submodule generated by linear combinations of links defined from local tangle modifications in oriented 3-balls. In quantum field theory the skein modules of M , or rather their duals, appear as modules of quantum observables. In link theory the skein module is the target of a *universal link invariant* satisfying given skein relations. Skein *algebras* have been studied in detail for cylinders over oriented surfaces, But not much is known about the structure of skein modules for general 3-manifolds. For an overview see [38].

We will construct various types of link invariants satisfying skein relations. Our construction is mostly algebraically and follows from the structure of the deformed 2-groupoid defined in section 3. Then some skein modules can be explicitly computed (e. g. HOMFLY skein modules for Lens spaces $L(p, 1)$). More often the existence of power series invariants of links in oriented compact 3-manifolds satisfying skein relations can be shown, see also [23]. This corresponds to the computation of completions of skein modules.

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Throughout let M be a compact oriented 3-manifold. For $k \geq 0$ let $\mathcal{L}[k]$ be the set of singular links with k ordered double-points. The elements of $\mathcal{L}[k]$ are the isotopy class of k -embeddings (immersions of circles in M with singularity precisely k double-points without tangencies). Thus $\mathcal{L}[0]$ is the usual set of oriented links in M (including the empty link). For $K_* \in \mathcal{L}[1]$ let K_\pm, K_0 be the usual Conway resolutions.

Let \mathcal{R} be a commutative ring with 1 and invertible elements q, v (which may very well be 1). Let $I \subset \mathcal{R}$ be some ideal. For each \mathcal{R} -module W let $W[[I]]$ denote the I -adic completion of W , i. e. the inverse limit of the system $W/I^i W$ for $i \geq 0$.

A map $\sigma : \mathcal{L}[1] \rightarrow \mathcal{R}\mathcal{L}[0]$ is called a *skein potential* (with respect to I) if the image of a j -component link is a linear combination of links with $< j$ components and arbitrary coefficients, and links with $\geq j$ components but coefficients in I .

We want to study skein relations of oriented links in M of the form $q^{-1}K_+ - qK_- = \sigma(K_*)$, and the following skein relation for framed oriented links: $q^{-1}K_+ - qK_- = \sigma(K_*)$, $K^{(+)} = qv^{-1}K$, where $K^{(+)}$ is the result of introducing a positive twist into a component of K .

In Chern Simons theory the *semi-classical* observables are the homotopy classes of oriented links in M . Thus it is natural to consider skein modules as *deformations* of free objects generated by homotopy classes of links in M . This is our main starting point.

Let $\mathfrak{b}[0]$ denote the set of homotopy classes of oriented links in M . We will choose *geometric models*, also called *standard links*, i. e. a map $\mathfrak{b}[0] \rightarrow \mathcal{L}[0]$ assigning to each homotopy class of link *some* oriented link with the given homotopy class. Let

$$\partial : \mathcal{L}[k] \rightarrow \mathcal{L}[k-1],$$

defined by $\partial(K_*) = q^{-2}K_+ - K_-$. ($*$ denotes the last double-point), be the *Jones boundary*. For $q = 1$ it is called the *Conway boundary*. Let

$$\mathcal{S}(\sigma) := \mathcal{R}\mathcal{L}[0]/\text{im}(\partial - q^{-1}\sigma)$$

be the *skein module* defined by σ .

In section 2 we will define the notion of *insensitive* skein potential. The usual oriented or framed oriented *local* skein relations are defined by insensitive skein potentials.

The main idea of the skein module is the *finite expansion* of links using the skein relation repeatedly. We will study expansions of links in terms of standard links by infinite applications of the skein relation. This is formalized in the following result.

Theorem 1. *Let σ be a skein potential and \mathfrak{s} be a choice of geometric models. Then there is defined a submodule*

$$U = U(\sigma, \mathfrak{s}) \subset \mathcal{R}(\mathfrak{b}[0])[[I]],$$

and the I -adic link invariant

$$\rho = \rho(\sigma, \mathfrak{s}) : \mathcal{L}[0] \rightarrow \mathcal{R}(\mathfrak{b}[0])[[I]]/U.$$

The submodule U is generated by (i) expansion of differentiability relations (see section 4), and (ii) expansion of elements in the image of a transversal string topology homomorphism (defined by σ). This homomorphism is defined on 1-dimensional

homology groups of the space of immersions in M . If σ is insensitive then U is determined only by (ii). The map ρ induces the module isomorphism

$$S(\sigma)[[I]] \cong \mathcal{R}\mathfrak{b}[[I]]/U.$$

The composition

$$\rho(\sigma, \mathfrak{s}) \circ \mathfrak{s} : \mathfrak{b}[0] \rightarrow \mathcal{R}(\mathfrak{b}[0])[[I]]/U$$

is the composition of the natural map $\mathfrak{b}[0] \rightarrow \mathcal{R}(\mathfrak{b}[0])[[I]]$ with the natural projection.

We will show that for insensitive σ the submodule U is completely determined by loops in the space of immersions defined from kink crossings, and by essential singular tori in M . This extends previous results of Kalfagianni [22], [23] and Kalfagianni and Lin [24].

The module $\mathcal{R}\mathfrak{b}[[I]]$ is an algebra with multiplication defined by formal multiplication of homotopy classes of maps. This multiplication induces on $S(\sigma)[[I]]$ the structure of a *skein algebra* if and only if U is an ideal in $\mathcal{R}\mathfrak{b}$. From the viewpoint of *physics*, if we cannot choose \mathfrak{s} such that $U(\sigma, \mathfrak{s})$ is an ideal then in a certain way the local skein relation σ does *not* provide the complete set of relations in *the algebra of loop observables* of the 3-manifold M .

Question 1. For which 3-manifolds M and skein relation σ can we choose \mathfrak{s} such that $U(\sigma, \mathfrak{s})$ is an ideal? How can we modify the ring \mathcal{R} to define a multiplication on $\mathcal{S}(\sigma)$, which *deforms* the multiplication of homotopy classes in M .

We will prove a result analogous to theorem 1 for *Vassiliev modules*. It is known that singular links provide a much clearer framework to study skein theory more generally.

Let $\mathcal{L} := \cup_{k \geq 0} \mathcal{L}[k]$ be the set of all singular links in M .

Let \mathcal{R} be as above and let $h \in \mathcal{R}$ be a *non-invertible element*. The *Vassiliev potential* $\sigma_V : \mathcal{L} \rightarrow \mathcal{I}\mathcal{L}$ defined by $\sigma_V(K_*) = hK_*$, where \mathcal{I} is the ideal generated by h . Then we can define the *infinite Vassiliev module*

$$\mathcal{S}(\sigma_V) := \mathcal{R}\mathcal{L}/\text{im}(\partial - q^{-1}\sigma).$$

Remark 1. Let $\mathcal{R} = \mathbb{Z}[h, q^{\pm}]$. The coefficient map $h, q \mapsto 1$ maps $\mathcal{S}(\sigma_V)$ onto the free abelian group $\mathbb{Z}\mathcal{L}[0]$. The images of the resulting relations $K_+ - K_- = K_*$ can be used inductively to eliminate all singular links with $k \geq 1$. Moreover,

$$\mathcal{L}[0] \rightarrow \mathcal{S}(\sigma_V) \rightarrow \mathbb{Z}\mathcal{L}[0]$$

is the inclusion of the basis. Thus the inclusion of $\mathcal{L}[0]$ into the skein module is injective.

Let $\mathfrak{b}[k]$ denote the set of homotopy classes of singular k -links. Then \mathfrak{b} is in one-to-one correspondence with a set of chord diagrams in M , with a certain ordering of the chords. We can choose geometric models \mathfrak{s} , i. e. links realizing the given homotopy classes. Let $\mathfrak{b} := \cup_{k \geq 0} \mathfrak{b}[k]$. In section 3 we will define a space of k -immersions in M for each $k \geq 1$.

Theorem 2. Let \mathfrak{s} be a given choice of geometric models. There is defined a submodule $U \subset \mathcal{R}\mathfrak{b}[[I]]$ and map

$$\rho(\sigma_V, \mathfrak{s}) : \mathcal{L} \rightarrow \mathcal{R}\mathfrak{b}[[h]]/U_V,$$

which induces the isomorphism

$$\mathcal{S}(\sigma_V)[[h]] \rightarrow \mathcal{R}\mathfrak{b}[[h]]/U_V.$$

The submodule U_V is generated by (i) expansions of differentiability relations, $4T$ -relations and tangency relations (defined from the local stratification of the space of immersions), and (ii) expansions of images of generalized string topology homomorphisms defined on the 1-dimensional homology of spaces of k -immersions. The composition

$$\rho(\sigma_V) \circ \mathfrak{s} : \mathcal{R}\mathfrak{b} \rightarrow \mathcal{R}\mathfrak{b}[[h]]/U_V$$

is the composition of the natural map $\mathfrak{b} \rightarrow \mathcal{R}\mathfrak{b}[[h]]$ with the natural projection.

The expansion map ρ in theorem 2 is defined on *all* singular links. The *local* differentiability and tangency relations can be subsumed in terms of the ordering of chords. But the geometric $4T$ -relations are very hard to control explicitly. This has been the obstruction for a topological construction of Kontsevitch's integral [5], [15] in the early 90's.

The result of theorem 2 easily generalizes to more general *skein potentials* for singular links $\mathcal{L} \rightarrow \mathcal{R}\mathcal{L}$ with a condition on components and coefficients in some ideal I as above.

Question 2. For which 3-manifolds M is the homomorphism

$$\mathcal{S}(\sigma_V) \rightarrow \mathcal{S}(\sigma_V)[[h]]$$

injective? Obviously $\rho(\sigma_V, \mathfrak{s})$ is a limit of Vassiliev invariants. Thus a positive answer would imply that links in M are classified by Vassiliev invariants.

From theorem 1 some explicit results concerning classical skein modules will be deduced. For $z, h \in \mathcal{R}$ non-invertible elements let $\sigma(K_*) = hK_0$ for a self-crossing and $\sigma(K_*) = zK_0$ for a crossing of different components. Then the skein module $\mathcal{J}(M)$ defined by oriented links with $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, z, h]$ and relations $q^{-1}K_+ - qK_- = \sigma(K_*)$ is called the *generalized Jones module* and has been considered in [44], [40]. The skein module $\mathcal{H}(M)$ is defined by framed oriented links with $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, v^{\pm 1}, z, h]$ and relations $q^{-1}K_+ - qK_- = \sigma(K_*)$, $K^{(+)} = qv^{-1}K$. This is the (*variant*) *HOMFLY skein module* (with split variables). For $z = h = s^{-1} - s$ this module has been discussed e. g. in [36], see also [13], [14] for results about of $S^2 \times S^1$ and connected sums. We will usually assume that the *vacuum* relations hold, i. e. $(q^{-1} - q)\emptyset = hU$ respectively $(v - v^{-1})\emptyset = hU$ for the unknot U and the empty link \emptyset .

The skein modules $\mathcal{J}(M)$ respectively $\mathcal{H}(M)$ are modules over the corresponding skein modules of the 3-ball, defined by disjoint union. Let \mathfrak{R} be the corresponding skein module of the 3-ball. It is known [39] and [44] that

$$\mathfrak{R} \cong \mathbb{Z}[q^{\pm 1}, z, h, \frac{q^{-1} - q}{h}]$$

respectively

$$\mathfrak{R} \cong \mathbb{Z}[q^{\pm 1}, v^{\pm 1}, z, h, \frac{v - v^{-1}}{h}].$$

Let \mathfrak{b}_0 be the set of homotopy classes of links in M without homotopically trivial components. Then there are defined \mathfrak{R} -homomorphisms \mathfrak{s} from the free modules

with basis \mathfrak{b}_0 into the modules $\mathcal{J}(M)$ respectively $\mathcal{H}(M)$. Then we have

$$\mathfrak{R}\mathfrak{b}_0 \cong S\mathfrak{R}\hat{\pi}^0,$$

where $\hat{\pi}^0$ is the set of non-trivial conjugacy classes of the fundamental group of M . Thus a *geometric model* \mathfrak{s} assigns to a monomial in $\hat{\pi}^0$ an oriented links with the homotopy classes of components given by the monomial. The geometric model is called *nice* if the following holds: If some element of $\hat{\pi}^0$ appears repeatedly in the sequence we assume that there exist self-isotopies of the representing link, which arbitrarily change the order of components with the same free homotopy classes. Moreover we assume that multiplication by $\frac{q^{-1}-q}{h}$ respectively $\frac{v-v^{-1}}{h}$ corresponds to adding some unknotted unlinked component to the corresponding standard link.

A 3-manifold M is called *atoroidal* respectively *aspherical* if each essential (i. e. the induced homomorphism of fundamental groups is injective) map of a torus $S^2 \times S^1$ respectively map of a 2-sphere S^2 in M is homotopic into the boundary of M .

Theorem 3. *Suppose that M is atoroidal and aspherical. Then the submodules of $\mathcal{J}(M)$ respectively $\mathcal{H}(M)$, which are generated by the image of a geometric model \mathfrak{s} , are isomorphic to $S\mathfrak{R}\hat{\pi}^0$.*

This can be used also to define invariants of links in rational functions satisfying the skein relations. Let \mathfrak{R}_0 denote the quotient field of the domain \mathfrak{R} , i. e. the field of fractions in \mathfrak{R} .

Theorem 4. *Suppose that M is atoroidal and aspherical. Then there are defined \mathfrak{R} -homomorphisms ι_β from $\mathcal{J}(M)$ respectively $\mathcal{H}(M)$ into \mathfrak{R}_0 for $\beta \in \mathfrak{b}_0$ satisfying $\iota_\beta \circ \mathfrak{s}(\beta') = \delta_{\beta, \beta'}$ for all $\beta, \beta' \in \mathfrak{b}$.*

Suppose that there is a nice geometric model map \mathfrak{s} , which is onto. Then we can explicitly compute the skein module. In particular the corollary follows from unpublished results of Lambropoulou and Przytycki [33].

Corollary 1. *Let $M = L(p, q)$ be a Lens space and $p \neq 0$. Then $\mathcal{J}(M)$ respectively $\mathcal{H}(M)$ is free over \mathfrak{R} , and isomorphic to $S\mathfrak{R}\hat{\pi}^0$*

In particular it follows now easily from Przytycki's universal coefficient theorem for skein modules that the usual variant HOMFLY module with $z = h =: s - s^{-1}$ is a free module over $\mathbb{Z}[q^{\pm 1}, v^{\pm 1}, \frac{v-v^{-1}}{s-s^{-1}}]$. The case for $p = 1$ has recently been solved by [Mr].

The simultaneous treatment of skein theory and Vassiliev theory is suggested by an observation due to Przytycki, see [41], also [25] and [6]. For $i \geq 0$ let $\mathcal{G}_i(M; \mathcal{R})$ be the quotient of $\mathcal{R}\mathcal{L}$ by the submodule generated by relations $K_* = K_+ - K_-$ for all $K_* \in \mathcal{L}$ and $*$ any double-point, and $K = 0$ for each singular link K with $> i$ double-points. Note that $\mathcal{G}_i(M; \mathcal{R})$ is generated by either $\mathcal{L}[0]$ or by the set of homotopy classes of singular k -links with $k \leq i$. The dual of $\mathcal{G}_i(M; \mathcal{R})$ is the module of \mathcal{R} -valued Vassiliev invariants of order $\leq i$ for links in M . In particular $\mathcal{G}_0(M; \mathcal{R})$ is dual to the module of type 0-invariants. This module is isomorphic to $S\mathfrak{R}\hat{\pi}$, where $\hat{\pi}$ is the set of conjugacy classes of the fundamental group of M . The module $\mathcal{G}_1(M; \mathcal{R})$ is the dual of the module of type 1-invariants, and has been discussed in [28]. It has been observed in [10], [11] that a generalized string topology homomorphism provides the indeterminacy of the universal type 1-invariant. The observation for homotopy skein modules has been made by the author in [16].

Let

$$\mathcal{G}(M; \mathcal{R}) := \lim_{\leftarrow} \mathcal{G}_i(M; \mathcal{R})$$

be the Vassiliev module of M . It has been observed by Przytycki that each skein relation defined by a skein potential σ (for $\partial(K_*) = K_+ - K_-$) defines an epimorphism

$$\mathcal{G}(M; \mathcal{R}) \rightarrow \mathcal{S}(\sigma)[[I]],$$

by mapping $K_* \in \mathcal{L}[1]$ to $K_+ - K_- - \sigma(K_*)$.

The relation between $\mathcal{G}(M; \mathcal{R})$ and $\mathcal{S}(\sigma_V)[[h]]$ is subtle. The fact that h is non-invertible is very important. Assume we are in the Conway case $q = 1$. For $i \geq 0$ let $\mathcal{W}_i(M; \mathcal{R})$ be the quotient of \mathcal{RL} by relations $K_+ - K_- - K_*$ for $*$ the *last* double-point, and $K = 0$ for singular links with $\geq i$ double-points. Applying the relation to K_{**} we get $K_{**} = K_{*+} - K_{*-}$. This is equal to $K_{+*} - K_{-*}$, which is the skein relation applied to another double-point, if and only if the differentiability relations hold. Note that $\mathcal{S}(\sigma_V)/h^i \mathcal{S}(\sigma_V)$ is *not* generated by links because we can only use $hK_* = K_+ - K_-$ to reduce the number of double-points. Roughly, this results in the fact that the order of chords in chord diagrams representing homotopy classes of singular links, can be changed only in corresponding powers of h .

Our main theorem 1 is a linearized version of a general construction for certain 2-groupoids. These 2-groupoids are constructed from certain 3-stage stratifications in spaces of k -immersions (immersions with at least k ordered double-points, see section 3 for the definition). We will construct a natural *deformation* and *quantization* of the usual homotopy 2-groupoid of the space of k -immersions of circles in M . It will be shown that the resulting algebraic 2-groupoid induces the usual skein and Vassiliev structures. The isotopy classes of links appear as the objects of a category, transversal deformations between links are the 1-morphisms of the category. The *topological* 2-groupoids constructed in this paper can be generalized to n -groupoids deforming and quantizing the usual homotopy n -groupoids of the spaces of k -immersions. It is the final goal to construct deformation structures of string topology in this setting, see [19]. In the language of 2-groupoids the *string topology* homomorphisms are a structure relating 2-morphisms and equivalence classes of 1-morphisms.

It will be important that the categorical structure above can be *linearized* similarly to the quantum invariant setting. Our viewpoint in this paper is much more general than necessary to deduce the explicit results above. But it is interesting to realize that *the structures inherent in skein modules and Vassiliev invariants, are a necessary consequence of the groupoid structure.*

The relation of the invariants defined in this paper with the Kontsevitch integral, Bott-Taubes invariants or Chern Simons polynomial invariants is unknown. In [31] there is *defined* a Kontsevitch type invariant based on the temporal gauge and the Feynman path integral. In the author's opinion this invariant is closest in spirit with the combinatorial invariants defined in this paper. This will be studied in future work.

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2. FORMAL SKEIN THEORY

For more details about category and higher category theory see [4], [27] and [35].

A 2-category is a triple $(\mathbf{ob}, \mathit{hom}, \mathit{mor}) = (\mathit{hom}_0, \mathit{hom}_1, \mathit{hom}_2)$, for which both $(\mathbf{ob}, \mathit{hom})$ and $(\mathit{hom}, \mathit{mor})$ are categories in the usual sense. More precisely, the following structures are given. For $x, y \in \mathbf{ob}$ there is defined $\mathit{hom}(x, y)$ with composition:

$$\circ : \mathit{hom}(x, y) \times \mathit{hom}(y, z) \rightarrow \mathit{hom}(x, z).$$

The elements in $\mathit{hom}(x, y)$ are called 1-morphisms. Composition is often abbreviated $u \circ v =: uv$. The morphisms between 1-morphisms are called 2-morphisms. The set of 2-morphisms from u to v is denoted $\mathit{mor}(u, v)$, and is non-empty only for *parallel* objects $u, v \in \mathit{hom}(x, y)$. There are two compositions of 2-morphisms. For $u, v, w \in \mathit{hom}(x, y)$ there is defined:

$$\circ_1 : \mathit{mor}(u, v) \times \mathit{mor}(v, w) \rightarrow \mathit{mor}(u, w).$$

For $x, y, z \in \mathbf{ob}$, $u_1, u_2 \in \mathit{hom}(x, y)$ and $v_1, v_2 \in \mathit{hom}(y, z)$ there is defined

$$\circ_2 : \mathit{mor}(u_1, u_2) \times \mathit{mor}(v_1, v_2) \rightarrow \mathit{mor}(v_1 \circ u_1, v_2 \circ u_2).$$

Recall the source and target maps $\mathit{hom} \rightarrow \mathbf{ob}$ and $\mathit{mor} \rightarrow \mathit{hom}$.

A 2-category in which all morphisms are equivalences is called a 2-groupoid. Thus for $u \in \mathit{hom}(x, y)$ there exists $v \in \mathit{hom}(y, x)$ such that both $u \circ v$ and $v \circ u$ are equivalent to the corresponding identity morphisms id . Equivalence means e. g. that $v \circ u$ and id are related by a 2-morphism, i. e. $\mathit{mor}(v \circ u, id) \neq \emptyset$. A 2-groupoid *with inverses* is a small 2-groupoid with a choice of $v := u^{-1} \in \mathit{hom}(y, x)$ for each $u \in \mathit{hom}(x, y)$. Recall that an n -category is small if the class of objects and all classes of morphisms are sets. In the following without mentioning *we will assume that all our 2-groupoids are small and with inverses*.

The composition of 1-morphisms is associative up to the action of 2-morphisms. The set of 2-morphisms $\mathit{mor}(u, u)$ is a group under \circ_1 . For $x \in \mathbf{ob}$ the set of equivalence classes of elements in $\mathit{hom}(x, x)$ under the action of mor is a group with composition defined by $[u] \cdot [v] := [u \circ v]$. (This is well-defined because of the composition \circ_2 .) Here $u, v \in \mathit{hom}(x, x)$ are equivalent if $\mathit{mor}(u, v) \neq \emptyset$.

Definition 1. A 2-category *with models* is a 2-category with a distinguished subset $\mathbf{m} \subset \mathbf{ob}$ such that for each $x \in \mathbf{ob}$ there exists $b \in \mathbf{m}$ such that $\mathit{hom}(x, b) \neq \emptyset$. If $\mathit{hom}(b_1, b_2) = \emptyset$ for all $b_1 \neq b_2$ then \mathbf{m} is called a set of *minimal models* of the 2-category.

For each set X let $F(X) =: F^{(0)}(X)$ be the free group generated by the elements of X . Let $F^{(j)}(X) := F(F^{(j-1)}(X))$ be defined inductively for $j \geq 1$. Let

$$\hat{F}(X) := \prod_{j \geq 0} F^{(j)}(X)$$

be the group defined by the infinite product. The group $\hat{F}(X)$ is not a free group but inverse limit of free groups.

Example 1. The group $\hat{F}(X)$ is always non-commutative for $X \neq \emptyset$. Let $X = \{*\}$. Then $F(X) = \mathbb{Z}$ but already $F^{(2)}(X) = F(\mathbb{Z})$ is the free group on an infinite number of generators. Consider the normal subgroup $2\mathbb{Z} \subset \mathbb{Z}$. Then $F(2\mathbb{Z}) \subset F(\mathbb{Z})$ is obviously not a normal subgroup.

There are natural inclusion maps $F^{(j-1)}(X) \subset F^{(j)}(X)$ defined by the inclusion of the basis. These maps are not homomorphisms because the product of the elements x_1, x_2 in $F^{(j-1)}$ maps to the basis element x_1x_2 . The inclusions combine to define the natural *shift map* $\widehat{sh} : \widehat{F}(X) \rightarrow \widehat{F}(X)$. Note that each map $Y \rightarrow F(X)$ comes with the usual extension, usually denoted by the same letter, $F(Y) \rightarrow F(X)$ from the universal property of the free group functor. Note that the extension of the inclusion $sh : F^{(j-1)}(X) \subset F^{(j)}(X)$ is the identity homomorphism. Now consider a homomorphism $h : F(Y) \rightarrow F(X)$. We can consider it as a map from the set $F(Y)$ into the group $F(X)$ and extend to the homomorphism $F(F(Y)) \rightarrow F(X)$. By composition with the inclusion $F(X) \subset F^{(2)}(X)$ we have defined the map $h_\bullet : F^{(2)}(Y) \rightarrow F^{(2)}(X)$. Note that there is also the homomorphism $F(h) : F^{(2)}(Y) \rightarrow F^{(2)}(X)$ defined by mapping a word $w_1 \dots w_r$ with $w_i \in F(Y)$ to $h(w_1) \dots h(w_r)$, which is different from h_\bullet .

Remark 2. The group $\widehat{F}(X)$ has a second natural product reminiscent of the multiplication of polynomials. In fact if $(a_i), (b_j)$ are two elements in $\widehat{F}(X)$ let (c_k) be the sequence defined by

$$c_k = \sum_{i+j=k} a_i b_j,$$

where the multiplication is performed in $F^{(k)}$ using the inclusion maps $F^{(i)} \subset F^{(k)}$ and $F^{(j)} \subset F^{(k)}$.

Moreover the shift map defines a natural action of the group $\widehat{F}(X)$ on the set $\widehat{F}(X)$ defined by $a \star b := a \cdot \widehat{sh}(b)$ where the multiplication on the right hand side is the usual multiplication in $\widehat{F}(X)$

There is a related natural construction. Note that the identity map from the set $F(X)$ into the group $F(X)$ extends to the homomorphism $F(F(X)) \rightarrow F(X)$. By iteration there are naturally defined homomorphisms $F^{(j)}(X) \rightarrow F(X)$ for each j . Thus we have defined the homomorphism:

$$c : \widehat{F}(X) \rightarrow \widetilde{F}(X) := \prod_{j=0}^{\infty} F(X).$$

Note that there is defined a natural shift homomorphism \widetilde{sh} on $\widetilde{F}(X)$ such that $c \circ \widehat{sh} = \widetilde{sh} \circ c$. The shift homomorphism \widetilde{sh} is a kind of *universal operator action* representing the multiplication by a fixed ring element in the commutative module case.

There is the unique natural *functor*

$$\mathfrak{d} : hom \rightarrow F(\mathfrak{ob})$$

defined by

$$\mathfrak{d}(u) = target(u)source(u)^{-1},$$

where $F(X)$ is considered as a monoid.

A *potential* of a 2-groupoid is a *contravariant functor*

$$\mathfrak{a} : hom \rightarrow F(\mathfrak{ob}).$$

We will use \mathfrak{a} to define inductively *power series expansions* of objects in terms of models. Note that \mathfrak{a} uniquely extends to homomorphisms

$$F^{(j)}(\mathfrak{a}) : F^{(j)}(hom) \rightarrow F^{(j+1)}(\mathfrak{ob})$$

for $j \geq 1$.

Theorem 5. *Let \mathcal{C} be a 2-groupoid with models $\mathfrak{m} \subset \mathfrak{ob}$. Let \mathfrak{a} be a potential. Then there exists a natural normal subgroup $\mathcal{A} \subset \hat{F}(\mathfrak{m})$, and the map*

$$\rho : \mathfrak{ob} \rightarrow \hat{F}(\mathfrak{m})/\mathcal{A},$$

such that the induced homomorphism:

$$\rho : \hat{F}(\mathfrak{ob}) \rightarrow \hat{F}(\mathfrak{m})/\mathcal{A}$$

satisfies the formal skein relation:

$$\rho(\mathfrak{d}(u)) = \widehat{sh}(\rho(\mathfrak{a}(u^{-1})))$$

for all $u \in \mathit{hom}_1$ and the induced map

$$\widehat{sh} : \hat{F}(\mathfrak{m})/\mathcal{A} \rightarrow \hat{F}(\mathfrak{m})/\mathcal{A}.$$

Proof. The normal subgroup \mathcal{A} and the homomorphism ρ is constructed from a sequence of normal subgroups $\mathcal{A}_j \subset F^{(j)}(\mathfrak{m})$ and homomorphisms $\rho_j : \mathfrak{ob} \rightarrow F^{(j)}(\mathfrak{m})/\mathcal{A}_j$. Let \mathcal{A}_0 be the trivial group. Define the map $\rho_0 : \mathfrak{ob} \rightarrow F^{(0)}(\mathfrak{m})$ by assigning to each $x \in \mathfrak{ob}$ its unique model $m \in \mathfrak{m}$. We define $\rho_1 : \mathfrak{ob} \rightarrow F^{(1)}(\mathfrak{m})/\mathcal{A}_1$ in the following way. Consider $u \in \mathit{hom}(x, m)$. There is defined $\mathfrak{a}(u) \in F(\mathfrak{ob})$. So we can apply the induced homomorphism

$$F(\rho_0) : F(\mathfrak{ob}) \rightarrow F^{(1)}(\mathfrak{m}).$$

Let $\lambda_1(u)$ be the resulting element. Define $\mathcal{A}_1 \subset F^{(1)}(\mathfrak{m})$ to be the normal subgroup generated by all elements $\lambda_1(u)\lambda_1(v)^{-1}$ for all $u, v \in \mathit{hom}(x, m)$. Then the image of $\lambda_1(u)$ in the quotient $F^{(1)}(\mathfrak{m})/\mathcal{A}_1$ does not depend on $u \in \mathit{hom}(x, m)$ but only on x . Next suppose that we have defined $\rho_j : \mathfrak{ob} \rightarrow F^{(j)}/\mathcal{A}_j$. Let ρ_j^1 be a lift of ρ_j to $F^{(j)}(\mathfrak{m})$. Note that the set of lifts is an orbit of \mathcal{A}_j , and we can write ρ_j^a for the lift differing from ρ_j^1 by the action of $a \in \mathcal{A}_j$. For $u \in \mathit{hom}(x, m)$ define $\lambda_{j+1}^a(u) \in F^{(j+1)}(\mathfrak{m})$ by application of

$$F(\rho_j^a) : F(\mathfrak{ob}) \rightarrow F(F^{(j)}(\mathfrak{m})) = F^{(j+1)}(\mathfrak{m})$$

to the element $\mathfrak{a}(u)$. Then define \mathcal{B}_{j+1} by the normal subgroup of $F^{(j+1)}(\mathfrak{m})$ defined by all elements

$$\lambda_{j+1}^{a_1}(u)(\lambda_{j+1}^{a_2}(v))^{-1}$$

for all $u, v \in \mathit{hom}(x, m)$ and all $a_1, a_2 \in \mathcal{A}_j$. Then the image $\rho_{j+1}(u)$ of $\lambda_{j+1}^a(u)$ in the quotient $F^{(j+1)}(\mathfrak{m})/\mathcal{B}_{j+1}$ does not depend on the choice of $a \in \mathcal{A}_j$ or $u \in \mathit{hom}(x, m)$ and ρ_{j+1} is well-defined. Finally define \mathcal{A}_{j+1} by the normal subgroup generated by \mathcal{B}_{j+1} and all elements of the form $\widehat{sh}(x^{-1})\widehat{sh}(xa)$ for all $x \in F^{(j)}(\mathfrak{m})$ and all $a \in \mathcal{A}_j$. Then obviously \widehat{sh} induces the map $\widehat{sh} : \hat{F}(\mathfrak{m})/\mathcal{A} \rightarrow \hat{F}(\mathfrak{m})/\mathcal{A}$. The second claim follows comparing $\rho(\mathit{target}(u))$ and $\rho(\mathit{source}(u))$. In the definition of ρ any morphism can be chosen so for a choice of $v \in \mathit{hom}(\mathit{target}(u), m)$ in the first case we can choose $vu \in \mathit{hom}(\mathit{source}(u), m)$ in the second case where m is the model for $\mathit{target}(u)$ and $\mathit{source}(u)$. Then the equality follows immediately from $\mathfrak{a}(vu) = \mathfrak{a}(u)\mathfrak{a}(v)$. \square

The subgroup \mathcal{A} depends inductively on all possible choices of morphisms in the sets $\mathit{hom}(x, m)$. This is difficult to describe in the most abstract setting, in

particular because of the lack of structure of the shift \widehat{sh} . Thus we consider the composition of ρ with the projection onto $\widetilde{F}(\mathfrak{m})$.

For each set X and subset $U \subset \widetilde{F}(X)$ let $(U)_{\widetilde{sh}}$ denote the smallest normal subgroup of $\widetilde{F}(X)$ containing all subsets $\widetilde{sh}^j(U)$ for all $j \geq 0$.

Theorem 6. *The map ρ induces the isomorphism*

$$\widetilde{\rho} : \widetilde{F}(\mathfrak{ob}) / (\mathfrak{d}(u)\widetilde{sh}(\mathfrak{a}(u^{-1}), u \in \text{hom})_{\widetilde{sh}}) \cong \widetilde{F}(\mathfrak{m}) / \widetilde{\mathcal{A}}.$$

Proof. The homomorphism c induces the epimorphism

$$\widehat{F}(\mathfrak{m}) / \mathcal{A} \rightarrow \widetilde{F}(\mathfrak{m}) / \widetilde{\mathcal{A}}$$

with $\widetilde{\mathcal{A}} := c(\mathcal{A})$. Thus by composition we have defined

$$\mathfrak{ob} \rightarrow \widetilde{F}(\mathfrak{m}) / \widetilde{\mathcal{A}}.$$

We extend this map to the homomorphism $\bar{\rho}$ defined on $\widetilde{F}(\mathfrak{ob}) = \prod_{j=0}^{\infty} F(\mathfrak{ob})$ by mapping the sequence (u_j) to $\prod_{j \geq 0} \widetilde{sh}^j(u_j)$. This is an infinite product but finite in each factor of $\widetilde{F}(\mathfrak{m})$, thus well-defined. Obviously

$$((\mathfrak{d}(u)\widetilde{sh}(\mathfrak{a}(u^{-1}), u \in \text{hom})) \subset \ker(\bar{\rho}).$$

Let $\widetilde{\rho}$ denote the homomorphism defined on the quotient. By construction $\bar{\rho}$ maps $\widetilde{F}(\mathfrak{m}) \subset \widetilde{F}(\mathfrak{ob})$ by projection onto $\widetilde{F}(\mathfrak{m}) / \widetilde{\mathcal{A}}$. This implies that $\widetilde{\rho}$ is onto. Moreover the composition $\widetilde{\mathcal{A}} \subset \widetilde{F}(\mathfrak{m}) \subset \widetilde{F}(\mathfrak{ob}) \rightarrow \widetilde{F}(\mathfrak{ob}) / (\mathfrak{d}(u)\widetilde{sh}(\mathfrak{a}(u^{-1}), u \in \text{hom})_{\widetilde{sh}}$ is the trivial homomorphism by the very construction of \mathcal{A} . This easily implies the injectivity of $\widetilde{\rho}$. \square

Remark 3. The group $\widetilde{F}(\mathfrak{ob}) / ((\mathfrak{d}(u)\widetilde{sh}(\mathfrak{a}(u^{-1}), u \in \text{hom}))$ is the completion of the *a-skein group* defined by

$$\mathcal{S}(\mathfrak{a}) := \bigoplus_{j \geq 0} F(\mathfrak{ob}) / (\mathfrak{d}(u)sh(\mathfrak{a}(u^{-1})))_{sh},$$

where sh is the shift homomorphism on the direct sum and $(\)_{sh}$ is defined as above.

The advantage of working with $\widetilde{F}(\mathfrak{m})$ is mostly because of the easier description of the indeterminacy subgroup \mathcal{A} resulting from the homomorphism property of \widetilde{sh} . Note that the contributions to $\widetilde{\mathcal{A}}_{j+1}$ resulting from lower orders now can be describe by $\widetilde{sh}(\mathcal{A}_j)$.

We will *split* the contributions to \mathcal{A} now in a natural way associated to the two origins of relations. Actually such a splitting is only possible in each degree but we can argue inductively. The idea is that, because of the 2-groupoid structure, we can consider all $u \in \text{hom}(m, m)$ with $\text{mor}(u, 1) \neq \emptyset$ and all those with $\text{mor}(u, 1) = \emptyset$ but considered up to the action of mor . This suffices to describe \mathcal{A}_1 and thus by induction all \mathcal{A}_j . Note that the change from $u : x \rightarrow m$ to $v : x \rightarrow m$ is provided only by uv^{-1} and there is $h \in \text{mor}(u, v)$ if and only if there is $h' \in \text{mor}(uv^{-1}, 1)$.

Recall that the functor $\mathfrak{d} : \text{hom} \rightarrow F(\mathfrak{ob})$ induces the homomorphism $\mathfrak{d} : F(\text{hom}) \rightarrow F(\mathfrak{ob})$ by the free group property. Also recall that $\rho_0 : \mathfrak{ob} \rightarrow F(\mathfrak{m})$ assigns to each object its model, and let ρ_0 also denote the extension to $F(\mathfrak{ob})$.

Definition 2. Let $R \subset \text{mor}$ be a subset. An potential \mathfrak{a} is called *insensitive with respect to R* if for each $m \in \mathfrak{m}$ and $r \in \text{hom}(m, m)$ such that there exists $r \in \text{mor}(u, 1) \cap M$, the following holds: (i) $\rho_0(\mathfrak{a}(u)) = 0$, and (ii) there exists $w \in F(\text{hom})$ such that $\mathfrak{d}(w) = \mathfrak{a}(u)$ and $\mathfrak{a}(w) = 1$. The potential is called *insensitive* if it is insensitive with respect to $R = \text{mor}$.

Theorem 7. Suppose that \mathfrak{a} is insensitive. Then $\tilde{\mathcal{A}}$ is the normal group generated by the images of maps

$$\text{hom}(m, m)/\text{mor} \rightarrow \tilde{F}(\mathfrak{m})$$

for all $m \in \mathfrak{m}$ and all their shifts. These maps are defined inductively by arbitrary expansion of elements in $\text{hom}(m, m)$ using the skein relation and following the idea of the proof of theorem 5. It suffices to consider the image of a generating set of $\text{hom}(m, m)/\text{mor}$.

Proof. Let $u \in \text{hom}(m, m)$ with $\text{mor}(u, 1) \neq \emptyset$. The contribution from u to \mathcal{A}_1 is provided by applying ρ_0 from the proof of theorem 4 (respectively its c -projection) to $\mathfrak{a}(u)$. This involves a choice of homomorphism w from $\mathfrak{a}(u)$ to some element of $F(\mathfrak{m})$, such that $\mathfrak{d}(w) = \mathfrak{a}(u)$. Note that because of (i) of insensitivity we know that the contribution in \mathcal{A}_1 vanishes. Moreover because of (ii) we know that the contributions in higher order vanish too up to indeterminacies in order ≥ 2 . (We also need that \mathcal{A}_0 is the trivial group). By induction it can be proved that the contributions can assumed to be vanishing in all orders, up to the indeterminacies resulting from the choices described by elements of $u \in \text{hom}(m, m)$ with $\text{mor}(u, 1) = \emptyset$. \square

Consider $h \in \text{mor}(u, v)$ with $u, v : m \rightarrow m$. Let

$$d(h) := \mathfrak{a}(uv^{-1}) \in F(\mathfrak{ob}).$$

Note that $d(h)$ actually only depends on its source and target. Let $\mathcal{D} \subset F(\mathfrak{ob})$ be the normal closure of the image of d . Then there is well-defined homomorphisms

$$\gamma : \text{hom}(m, m)/\text{mor} \rightarrow F(\mathfrak{ob})/\mathcal{D},$$

which assigns to $u : m \rightarrow m$ the element $\mathfrak{a}(u)$. Here we use the group structure on $\text{hom}(m, m)/\text{mor}$ given by $u \cdot v := vu$, where on the right hand side the operation is composition. The homomorphism γ seems to be a kind of *categorical* version of a string topology homomorphisms of Chas and Sullivan. This will be discussed in [19].

Next we will linearize the above structures.

Definition 3. A *model complexity* is a map

$$\mathfrak{c} : \mathfrak{m} \rightarrow \mathbb{N},$$

where \mathbb{N} is the set of non-negative integers.

If \mathfrak{m} is minimal then the model complexity first extends to \mathfrak{ob} by $\mathfrak{c}(x) = \mathfrak{c}(m)$, where m is the unique model of x . Then extend to hom by defining $\mathfrak{c}(u) = \mathfrak{c}(\text{target}(u)) = \mathfrak{c}(\text{source}(u))$. Finally define $\mathfrak{c}(h) = \mathfrak{c}(u)$ for each 2-homomorphism $h : u \rightarrow v$. Note that $\mathfrak{c}(u) = \mathfrak{c}(v)$. There is always defined the trivial complexity, which is a constant. If there is no explicit mentioning of a complexity then we will assume that we work with the trivial complexity. Let $\mathfrak{ob}(j) := \mathfrak{c}^{-1}(j)$ and similarly for hom and mor .

Definition 4. Let $R \supset \mathbb{Z}$ be a commutative ring with 1.

(i) A *linear potential* is a contravariant functor:

$$\mathbf{a} : \mathit{hom} \rightarrow R\mathfrak{ob},$$

where $R\mathfrak{ob}$ is considered as a groupoid (abelian group), and hom is the category with objects \mathfrak{ob} and morphisms hom .

(ii) Let $I \subset R$ be an ideal and \mathfrak{c} be a model complexity. Then \mathbf{a} is a linear potential with respect to I if it maps $\mathit{hom}(j)$ into the direct sum of

$$R\left(\bigcup_{i < j} \mathfrak{ob}(i)\right) \oplus I\left(\bigcup_{j \leq i \leq \ell} \mathfrak{ob}(i)\right)$$

for each $j \geq 0$ and a fixed number ℓ independent of j .

Let $\partial : R(\mathit{hom}) \rightarrow R\mathfrak{ob}$ be defined by

$$\partial(u) = \mathit{target}(u) - \mathit{source}(u).$$

For a trivial model complexity we have a linear potential with respect to I if it maps *all* homomorphisms into $I(\mathfrak{ob})$.

Now replace in the constructions above for each set X the group product $\tilde{F}(X)$ by the infinite product of free modules

$$\prod_{j \geq 0} I^j X.$$

Replace the potential by the linear potential. Then the results above admit analogs in the commutative framework of R -modules. Note that

$$\prod_{i \geq 0} I^i X \cong RX[[I]].$$

The isomorphism is defined from the sequence of homomorphisms mapping $(u_i) \in \prod_{i \geq 0} I^i X$ to $\sum_{i \geq 0} u_i \in RX/I^j X$ for $j \geq 0$. This is compatible with projections and thus defines a homomorphism into the inverse limit, which is easily seen to be an isomorphism. Similarly we call \mathbf{a} insensitive if for each $u \in \mathit{hom}(m, m)$ with $\mathit{mor}(u, 1) \neq \emptyset$ we have that (i) $\rho_0(\mathit{l}(u)) = 0$ and (ii) there exists $w \in \mathit{hom}$ such that $\mathbf{a}(u) = \mathfrak{d}(w)$ and $\mathbf{a}(w) = 0$.

Example 2. Let R be a commutative ring with 1 and let $I = (h)$. Let X be a set. Let \mathfrak{c} be the trivial complexity. Then there exists the natural *linearization functor*

$$\tilde{F}(X) \rightarrow RX[[I]]$$

by defining it on the i -th factor of $\tilde{F}(X)$ such that $x \in X$ maps to $h^i x \in I^i X$. Note that this map factors through $\mathbb{Z}(X)$ on each factor. By composition with the linearization functor a potential becomes linear.

Theorem 8. Let \mathbf{a} be a linear potential with respect to I and model complexity \mathfrak{c} . Suppose that \mathbf{a} is insensitive. Then there exist a map

$$\mathfrak{ob} \rightarrow R\mathfrak{m}/\mathcal{I},$$

where the submodule \mathcal{I} is generated by the images and I -translates of generators under maps

$$\mathit{hom}(m, m)/\mathit{hom}_2 \rightarrow R\mathfrak{m}[[I]] \cong \prod_{i=0}^{\infty} I^i \mathfrak{m}.$$

Moreover there is an induced homomorphism:

$$R\mathfrak{ob}/(\partial(u) - \mathfrak{a}(u), u \in \text{hom}) \cong R\mathfrak{m}[[I]]/\mathfrak{I},$$

and an induced isomorphism

$$(R\mathfrak{ob}/(\partial(u) - \mathfrak{a}(u), u \in \text{hom}))[[I]] \cong R\mathfrak{m}[[I]]/\mathfrak{I}.$$

Proof. If \mathfrak{c} is a trivial model complexity then the proof is analogous to the proofs of the similar results before. Otherwise consider the subset $\mathfrak{ob}(i) \subset \mathfrak{ob}$. Then we will change the construction of theorem 4 into the k -th factor in $\prod_i R\mathfrak{m}$. Let ℓ be the nonnegative integer defined in definition 4. Then for a given complexity i , all elements of $\mathfrak{ob}(i)$ map into $I^{k-j}\mathfrak{m}$ for some $j \leq i + (k-j)\ell$. This can easily be proved by induction. Thus $j(\ell + 1) \leq i + k\ell$ and

$$k - j \geq k - \frac{i + k\ell}{\ell + 1} =: \phi(i, \ell, k).$$

The sequence of numbers $\phi(i, \ell, k)$ converges to infinity for k to infinity. (Essentially we can replace the inverse limit over k by an inverse limit over $\phi(i, \ell, k)$, see the remark below). Now one can construct submodules \mathfrak{I}_j and homomorphisms

$$\mathfrak{ob} \rightarrow \bigoplus_{j \leq k} R\mathfrak{m}/\mathfrak{I}_k =: \mathfrak{M}_k$$

for each fixed i, k . We may have to apply the inductive procedure of theorem 4 more than k times to define the image in \mathfrak{M}_k . In fact just apply it k' times with $\phi(i, \ell, k') \geq k$, but do the computation mod I^{k+1} . \square

Remark 4. It is possible to construct submodules $\mathcal{C}_k \subset R\mathfrak{m}/I^k\mathfrak{m}$ and compatible maps

$$\mathfrak{ob} \rightarrow (R\mathfrak{m}/I^k\mathfrak{m})/\mathcal{C}_k$$

directly in the following way. The maps will be constructed by application of the main procedure from theorem 4, i. e. by replacing objects by model objects, and elements in the image of \mathfrak{a} based on the *skein relation* $\partial(u) = \mathfrak{a}(u)$ for $u \in \text{hom}$. More precisely construct maps on $\mathfrak{ob}(i)$ for each k by induction on the complexity i . Then we can assume that we have the expansion in $R\mathfrak{ob}$:

$$x = \sum_j a_j m_j + b + r,$$

with $a_j \in R$, $m_j \in \mathfrak{m}$ and r a linear combination of skein relations. Let $b = \sum_n b_n x_n$ with $b_n \in I^k$ and $x_n \in \mathfrak{ob}$. Note that we can assume inductively that each object in the linear combination b has complexity at most $i + \ell k$. Now apply the skein relations to x_n . This will give rise to a linear combination containing models, elements of the form ay with $a \in I$ and $y \in \mathfrak{ob}$ and elements of the form $a'y'$ with $a' \in R$ but $c(y') \leq i + \ell - 1$. Thus after at most $\ell k + 1$ applications of the skein relation we will have written x as a linear combination

$$x = \sum_j a'_j m'_j + b + b' + r',$$

with $b \in I^{k+1}\mathfrak{ob}$, $c(b') < i$ and r' a linear combination of skein relations. Now we can apply the inductive assumption to expand b' in the form we want, and finally get an expansion we intend. The indeterminacies result from all possible choices in the procedure of applications of the skein relation.

We need one more definition:

Definition 5. We say that the 2-groupoid is *free on a set of elementary homomorphisms* S if the set hom is in one-to-one correspondence with the *subset* of $F(S)$ defined by the products of composable words in S^\pm such that the product of words corresponds multiplication in the free group. In this case we write $hom = F(S, \circ)$.

3. MAPPING SPACES AND TRANSVERSAL GROUPOIDS

Let M be an oriented compact 3-manifold. Let $Diff(M)$ be the group of diffeomorphisms of M , which are isotopic to the identity. Let $iso(M)$ be the space of paths $\gamma : I \rightarrow Diff(M)$ with $\gamma(0) = id$. (There is the fibration

$$\Omega Diff(M) \rightarrow iso(M) \xrightarrow{ev} Diff(M),$$

with $ev(\gamma) = \gamma(1)$, which should be of interest in the understanding of the 2-groupoids of this section.)

In this section we describe the topological 2-groupoids used in this paper. Our approach generalizes to spaces X with a suitable manifold stratification in codimensions $\leq n$ (generalizing spaces of immersions of circles into a 3-manifold), see [19]. In fact the easiest application here is to spaces of immersions of circles into higher dimensional manifolds.

The idea is to replace the fundamental n -groupoid by a subgroupoid consisting of maps transversal with respect to the stratification. The resulting n -groupoid is *quantized* with respect to certain morphisms, i. e. we form a quotient groupoid in which these morphisms are identities. For $n = 2$ and X a space of immersions of circles into the 3-manifold M , the quantization is with respect to 1-morphisms, which are isotopies (traces of the action of $Diff(M)$). We will discuss the general set-up in [K]. Here we only describe the resulting 2-groupoid. The reader will see that forming the quotient groupoid is a delicate problem on its own.

Let $\widetilde{imm}(j)$ be the set of immersions $\cup_j S^1 \rightarrow M$ with all immersions on the small diagonals deleted (i. e. immersions with identical components). It is known that $imm(j)$ is an infinite dimensional manifold modeled on an inverse limit of Hilbert spaces [7], chapter 3.

Note that the diagonals form a subset of infinite codimension which will be ignored in most discussions. The symmetric group Σ_j acts on $\widetilde{imm}(j)$ by permuting the circles in the domain. Let $imm(j)$ be the quotient manifold.

For each space X let $C_i(X)$ denote the space of ordered configurations of i points in X . An ordered i -configuration is just an embedding of the space $*_1 \cup \dots \cup *_i \hookrightarrow X$. Let $k \geq 1$. Consider the evaluation map

$$ev : \widetilde{imm}(j) \times C_{2k}(\cup_j S^1) \rightarrow M^{2k}$$

defined by $ev(f, x) := f(x)$. Let $\Delta \subset M^{2k}$ be the submanifold of those points $(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$ with $x_i = y_i$ for $i = 1, \dots, k$ but $x_i \neq x_j$ for $i \neq j$. Note that Δ is a locally closed submanifold of M^{2k} , and is diffeomorphic to the configuration space $C_k(M)$ of ordered k -configurations in M .

Let $\widetilde{imm}[k](j) := ev^{-1}(\Delta)$. The symmetric group acts on $\widetilde{imm}(j) \times C_{2k}(\cup_j S^1)$ by simultaneously permuting the circles in the domain of immersions and the circles in the codomain of the embeddings in $C_{2k}(\cup_j S^1)$. The group \mathbb{Z}_2^k acts by permuting

the points in the domain of embeddings $*_1 \cup \dots *_k \rightarrow \cup_j S^1$. Let $imm[k](j)$ be the quotient by the actions. (Again we may delete diagonals as above to have a free action.)

Now for $k \geq 0$ let

$$imm[k] := \bigcup_{j \geq 0} imm[k](j)$$

be the space of k -immersions in M . Note that a k -immersion always has k distinguished *ordered* double-points. The space $\widetilde{imm}[k](j)$ behaves like a submanifold of $\widetilde{imm}(j) \times C_{2k}(\cup_j S^1)$ of codimension k . This can easily be proved *in low codimension* from the singularity stratification on imm and the observations below, or by using the finite dimensional model of Vassiliev, see Appendix. Alternatively we can refer to the tangential description of $imm(1)$ given in [7], p. 111, from which the manifold structure can easily be deduced.

Remark 5. (i) The spaces $imm[k](j)$ are related with spaces of mappings of graphs into M . In fact, one can use the ordering of the configuration space coordinates, orientation and minimal distance between the $2k$ points in $\cup_j S^1$ to define a map from $\cup_j S^1$ onto oriented standard graphs, such that immersions with the given configuration factor through mappings of graphs. This can be done continuously over $imm[k](j)$. The graphs are equipped with an orientation around the vertices and basepoints on components away from the vertices. This will define a continuous mapping from $imm[k](j)$ onto a space of mappings of graphs into M . Using length parameters on edges of the graphs it is possible to construct some inverse map.
 (ii) The spaces $imm[k]$ may also be related with certain spaces of colored cacti, see [47].

Let $emb[k] \subset imm[k]$ be the subset representing immersions with precisely k -double-points without tangencies. These immersions are called k -embeddings.

For $k \geq 0$ let $\mathcal{L}[k]$ denote the set of *isotopy* classes of k -embeddings. Here two k -embeddings are isotopic if there is a continuous path in $emb[k]$ joining the two k -embeddings.

Then $\mathcal{L}[0]$ is the usual set of isotopy classes of oriented embeddings of circles also called oriented links in M . The elements of $\mathcal{L}[k]$ in general are called oriented k -links in M .

Let $imm[k, 1]$ denote the subspace of elements in $imm[k]$ representing immersions with precisely $k + 1$ double-points without tangencies.

Lemma 1. $imm[k, 1] \subset imm[k]$ is naturally homeomorphic to the space $emb[k + 1]$.

Proof. There is a natural ordering of the double-points by putting the distinguished double-point in the last place. Thus the sets are in one-to-one correspondence, which is easily seen to be a homeomorphism. \square

Let $imm[k, 2]$ denote the union of the set of elements in $imm[k]$ consisting of (i) immersions with $k + 2$ double-points without tangency, (ii) immersions with $k - 1$ double points without tangency and a triple point, and (iii) immersions with k double-points, $k - 1$ of which without tangency and a tangency at one of the double-points. Note that the immersions with (i) in $imm[k, 2]$ do have an ordering of double-points up to permutations of the last two double-points. Thus there is a natural 2-fold covering map $emb[k + 2] \rightarrow imm[k, 2]_{\bullet}$, where $imm[k, 2]_{\bullet} \subset imm[k, 2]$ is the subspace satisfying (i). In general let $imm[k, j]_{\bullet} \subset imm[k]$ be the subspace

represented by immersions, whose singularities are the k ordered double-points without tangency, and j additional double-points without tangency.

If we replace in all the definitions above the spaces $\widetilde{imm}(j)$ by spaces $\widetilde{map}(j)$ of differentiable maps $\cup_j S^1 \rightarrow M$ we still can define in the same way spaces $map[k]$ of differentiable maps with k double-points, and possibly further singularities.

We will now describe the sequence of 2-groupoids $\mathcal{C}[k]$ for $k \geq 0$. Skein theory and Vassiliev theory should be considered as a construction on these groupoids.

The set of objects of $\mathcal{C}[k]$ will be the set $\mathcal{L}[k]$ of singular links with k double points, i. e. isotopy classes ($Diff(M)$ -orbits) of immersions of circles in M with precisely k double points without tangencies.

The 1-morphisms in the category $\mathcal{C}[k]$ will be the *isotopy classes of transversal paths in $imm[k]$* . A path $\gamma : I \rightarrow imm[k]$ is *transversal* if $\gamma(t) \in emb[k]$ except for finitely many t where $\gamma(t)$ maps to an immersion with precisely $k+1$ double points without tangencies. Moreover, in the neighbourhood of such t the image of $\gamma(s)$ runs through a standard crossing change in some oriented 3-ball in M . It is well-known that any homotopy between two elements of $emb[k]$ can be approximated by a transversal homotopy. The set of points in $imm[k]$ with a tangency at one of the k double points has codimension 2 in $imm[k]$, also compare [34], 3.2.

Next we define *isotopy* of transversal paths. Let $\chi : I \rightarrow iso(M)$ be a *piecewise* continuous path. Assume that $t \mapsto \chi(t)(1) \in Diff(M)$ is continuous. Note that $\chi(t)(0) = id$ for all t . The *adjoint* of χ is the map $\chi' : I \times I \rightarrow Diff(M)$ defined by $\chi'(t, t') \rightarrow \chi(t)(t')$. Then χ acts as a *deformation* $I \times I \rightarrow imm[k]$ of γ by

$$(t, t') \mapsto \chi'(t, t') \circ \gamma(t).$$

For a given transversal path γ we assume that χ only jumps at parameters t away from the singular parameters of γ . We assume that χ is a jump function in the following way. At a jump parameter t the limit of the isotopies from the left or right is different from the isotopy at t . But we always assume continuity from left or right. Then we say that χ defines the *isotopy* $\gamma \rightarrow \gamma'$ with $\gamma'(t) = \chi(t)(1) \circ \gamma(t)$. The singular parameters of γ' and γ coincide. In this way isotopy can change γ between singular parameters by arbitrary loops in $emb[k]$. We will allow a second class of isotopies of a more trivial nature. Two transversal paths γ, γ' are also isotopic if there is a transversal homotopy $I \times I \rightarrow emb[k] \cup imm[k, 1]$ restricting to γ respectively γ' on $I \times \{0\}$ respectively $I \times \{1\}$. Transversality of the homotopy means that the preimage of the set $imm[k] \setminus emb[k]$ is a 1-dimensional manifold properly embedded in M .

Note that composition of 1-morphisms is well-defined. In fact, let $\gamma : K_1 \rightarrow K_2$ and $\gamma' : K_2 \rightarrow K_3$ are homomorphisms for singular links K_i , $i = 1, 2, 3$. Here we choose representative paths γ, γ' between corresponding representing immersions. We can choose any isotopy between representatives $\gamma(1), \gamma'(0)$ of K_2 , and take the usual composition of paths. The result of the composition does not depend on the choice of isotopy up to isotopy of transversal paths. The composition \circ of transversal paths is associative because isotopy includes reparametrizations of the interval. But in general $\gamma \circ \gamma^{-1}$ is not isotopic to the constant path because the embedding of the singular parameters in I is well-defined up to isotopy.

Finally the 2-morphisms between any two isotopy classes of transversal paths will be defined by *homotopy classes of transversal homotopies* between representing paths in $imm[k]$. The appropriate notion of homotopy class is defined below. There are obvious ways of horizontal and vertical composition of maps $I \times I \rightarrow imm[k]$ with certain boundary restrictions coinciding (corresponding to \circ_2 and \circ_1). A *transversal homotopy* is the result of *horizontal and vertical composition* of

(i) continuous maps

$$I \times I \rightarrow imm[k],$$

which are constant on $I \times \{0, 1\}$, and

(ii) maps $I \times I \rightarrow imm[k]$, which are adjoint to piecewise continuous paths $I \rightarrow iso(M)$ as defined above. (in general this is not constant on $I \times \{0, 1\}$, but the restriction to $I \times \{0, 1\}$ is two continuous paths in emb), and

(iii) transversal maps $I \times I \rightarrow emb[k] \cup imm[k, 1]$.

The transversality of 2-morphisms will be in the spirit of Lin [34]. We can assume that the preimage of the set of those elements in $imm[k]$, which are not in $emb[k]$, is a graph with vertices of valence 1, 2 or 4. All vertices of valence 1 will be contained in the boundary $\{0, 1\} \times I$ and be mapped to immersions with precisely $k + 1$ double-points without tangencies. All the open edges of the graph are mapped to immersions with precisely $k + 1$ double-points without tangencies. The vertices of valence 4 or 1 are contained in the interior of $I \times I$. Those of valence 4 are mapped to either (i) immersions with $k + 2$ double-points without tangency, or (ii) immersions with one transversal triple point (with a distinguished branch) and $k - 1$ double points without tangency. The vertices of valence 2 are mapped to immersions with k double points but with a tangency at one of those double-points. Note that for $k = 0$ both the tangency contributions and triple points vanish.

Now we explain homotopy of transversal homotopies. Note that a 2-morphism has source and target *isotopy* classes of transverse paths. We will generate homotopy of 2-morphisms by homotopy of (i), (ii) and compositions \circ_1, \circ_2 . A homotopy of (i) is a continuous map

$$(I \times I) \times I \rightarrow imm[k],$$

which is constant on $(I \times \{0, 1\}) \times I$. Finally we *define* any two transversal homotopies of type (ii) mapping the transversal path γ to the transversal path γ' to be homotopic to the identity on the class of γ . It is now easy to see that the set of 2-morphisms mor is well-defined. Note that the definitions are straightforward in order to have well-defined compositions operations on the quotient groupoid.

Lemma 2. *Each homotopy between transversal paths joining two k -embeddings can be approximated relative to the boundary by a transversal homotopy.*

Proof. We choose a finite dimensional model for the space of immersions, see [2],[45] and the appendix. It can be assumed that evaluation maps restricted to $ev^{-1}(\Delta) \subset \widetilde{imm}(j) \times C_{2k}(\cup_j S^1)$ corresponding to higher order singularities and jet singularities are transversal. Then the result follows from known local models of the singularity strata and their codimensions in these spaces. Obviously the open stratum in $imm[k]$ is $emb[k]$, the codimension-1 stratum consists of immersions with precisely $k + 1$ ordered double-points without tangency. The codimension 2-stratum is constructed from the codimension-1 generic degeneracies in the limit set of the immersions in the codimension-1 stratum. \square

Remark 6. (i) Lin's transversality results [34] are the piecewise linear versions of the results of the lemma, see also the results by Stanford [43], Bar-Natan and Stoimenov [5] and Hutchings [15].

(ii) The result of the lemma holds for any map of a surface $F \rightarrow \text{imm}[k]$ with given transversal map on the boundary ∂F .

Definition 6. The *graded 2-groupoid*

$$\mathcal{C}(M) := \bigcup_{k \geq 0} \mathcal{C}(k)$$

is called the *Vassiliev groupoid* of M .

The following two basic results describe the structure of the 2-groupoids $\mathcal{C}[k]$. The first theorem is immediate from the definitions. The proof of the second one follows from lemma 1.

Theorem 9. *For each $k \geq 0$ the morphism set $\text{hom}[k]$ is \circ -generated by the elementary morphisms $K_+ \rightarrow K_-$ defined by crossing changes, and their inverses $K_- \rightarrow K_+$, for all $K_* \in \mathcal{L}[k+1]$ with $*$ indicating the last double-point. By identification of $\mathcal{L}[k+1]$ with the elementary morphisms in $\mathcal{C}[k]$ we have*

$$\text{hom}[k] = F(\mathcal{L}[k+1], \circ).$$

□

Note that for any $K \in \mathcal{L}[k]$ there is the unique 1-morphism 1, which is represented by *any* isotopy (path in $\text{emb}[k]$) between representing embeddings of K .

We describe resolutions of triple points. Consider immersions in $\text{imm}[k]$ with precisely $k-1$ double points without tangencies and a transverse triple point. There is a distinguished branch and an ordering of the remaining two branches at the triple point. Let $\mathcal{T}[k]$ denote the set of isotopy classes of these immersions. Let $K \in \mathcal{T}[k]$. We will define resolutions $K_{\pm}^{1/2} \in \mathcal{L}[k+1]$ in the following way: $K_{\pm}^{1/2}$ ist the positive respectively negative resolution of the double point, which appears when the first respectively second branch is ignored and only the remaining and the distinguished branch are considered to intersect at the triple point. The geometric $4T$ -relation will be built with these singular links. See [Mu] for the geometric description of geometric $4T$ -relations.

Theorem 10. *The morphism set $\text{mor}[k]$ is generated via compositions \circ_1 and \circ_2 by elementary 2-morphisms of the following form:*

- (i) *morphisms $K_{-*}^{-1} K_{*+}^{-1} K_{+*} K_{*-} \rightarrow 1$ (1 is the 1-morphism for K_{--}) defined for each $K_{**} \in \mathcal{L}[k+2]$. (differentiability relations)*
- (ii) *for $k > 0$, morphisms $(K_+^1)^{-1} (K_+^2)^{-1} K_-^1 K_-^2 \rightarrow 1$ defined for each $K \in \mathcal{T}[k]$ (geometric $4T$ -relation).*
- (iii) *for $k > 0$, morphisms $K \rightarrow K'$ for $K, K' \in \mathcal{L}[k+1]$ defined by changing the order of the two double-points appearing in the natural two-point resolution of a tangency.*

Proof. The result is immediate from the definitions and lemma 1. □

Remark 7. It is possible to define a slight variation of the 2-groupoid \mathcal{C} by changing the definition of isotopy of 1-morphisms given above in the following way. Instead of allowing arbitrary transversal homotopies in $\text{emb}[k] \cup \text{imm}[k, 1]$ only allow homotopies defined by reparametrizations of the interval (use homotopies from *id* into

monotone functions $I \rightarrow I$). Then of course we have to introduce the isotopies of the above form into the 2-morphisms. In this case the 1-morphisms $hom[k]$ will be identified with the *monoid* generated by $\mathcal{L}[k+1]^{\pm 1}$. Then in the statement of theorem 9 we have introduced $uu^{-1} \leftrightarrow 1$.

We can construct a set of models \mathfrak{m} for the category \mathcal{C} by choosing an element in $emb[k]$ for each path component of the spaces $imm[k]$ and $k \geq 0$. Note that the set of path-components of $imm[k]$ is the set of homotopy classes of singular k -links. $\mathfrak{b} = \cup_{k \geq 0} \mathfrak{b}[k]$. Note that $\mathfrak{b}[0]$ is in one-to-one correspondence with the set of monomials in $\hat{\pi}$, where $\hat{\pi}$ is the set of conjugacy classes of $\pi_1(M)$.

For $k \geq 1$, the homotopy classes of singular k -links are in one-to-one correspondence with the set of chord diagrams in M with k chords. These are usual chord diagrams equipped with free homotopy classes of maps into M assigned to each component of the complement of the set of endpoints of chords in $\cup_j S^1$. We will have an ordering of the chords for each representative chord diagram. Note that it is possible that chord diagrams with different orderings are homotopic, depending on the mappings into M . For example in S^3 this is always true and chord diagrams do not carry any ordering of the chords.

The differentiability relations impose *commutativity on the level of models*. This means that indeterminacies of expansions defined by abstract skein potentials in $\tilde{F}(\mathfrak{ob})$ in section 2 contain the commutators of \mathfrak{ob} . Thus it suffices to consider skein potentials like in the usual skein or Vassiliev theory.

Next we will define a certain 2-groupoid \mathcal{B} from suitable bordism classes of mappings of i -dimensional manifolds into imm for $i = 0, 1, 2$. In this category the 1-morphisms are *linearized* in a certain way, as is suggested by the notion of skein potential in a commutative ring with 1. The precise definition of this 2-groupoid, called the *skein groupoid* turns out to be quite interesting and subtle on its own. In the following note that bordism and homology coincide in dimensions ≤ 2 , see [26].

The objects of $\mathcal{B}[k]$ are the same as in $\mathcal{C}[k]$, i. e. the isotopy classes of embeddings into $emb[k]$.

The 1-morphisms $x \rightarrow y$, for $x, y \in \mathcal{L}[k]$, are the oriented bordism classes u of maps of oriented compact 1-manifolds $W \rightarrow emb[k] \cup imm[k, 1]$, i. e. elements of $H_1(emb[k] \cup imm[k, 1], emb[k])$ such that $\partial(u) = y - x$, where

$$\partial : H_1(emb[k] \cup imm[k], emb[k]) \rightarrow H_0(emb[k])$$

is the usual boundary operator. Note that this is well-defined since $H_0(emb)$ is the free abelian group on $\mathcal{L}[k]$. Thus the set of 1-morphisms hom in \mathcal{B} is a certain *subset* of the homology group. Note that W can be represented by a map

$$(I \cup \cup_i S^1, \partial I) \rightarrow (emb[k] \cup imm[k], emb[k]).$$

There can be *arbitrary* maps of closed components into $emb[k] \cup imm[k, 1]$. Note that a representative map may very well contain other component maps $I \rightarrow emb[k] \cup imm[k, 1]$. But the homological boundary of all these components vanishes in $emb[k]$.

The 2-morphisms $u_1 \rightarrow u_2$ for 1-morphisms $u_i : x \rightarrow y$ and $i = 1, 2$, are more difficult to describe. See [48] for a description of the general bordism set-up used

here. We consider suitable *bordism classes* of quadruples of maps ($j = 1, 2$)

$$g : (F, \partial_j F, \partial_1 F \cap \partial_2 F) \rightarrow (imm[k], emb[k] \cup imm[k, 1], emb[k]),$$

where F is an oriented compact surface with boundary $\partial F = \partial_1 F \cup \partial_2 F$ such that $\partial_1 F \cap \partial_2 F$ is a disjoint union of two points. We require that $g[(\partial_j F)]$ represents $u_j \in H_1(imm[k] \cup emb[k], emb[k])$ for $j = 1, 2$. Suppose that we have given two quadruples g_i , $i = 1, 2$, as above:

$$g_i : (F_i, \partial_j F_i, \partial_1 F_i \cap \partial_2 F_i) \rightarrow (imm[k], emb[k] \cup imm[k, 1], emb[k]).$$

A bordism from g_1 to g_2 as above is a quadruple of maps

$$G : (W, \partial_j W, \partial_1 W \cap \partial_2 W) \rightarrow (imm[k], emb[k] \cup imm[k, 1], emb[k]),$$

where W is an oriented compact 3-manifold with corners. We have that $\partial_j(W) \subset \partial W$ is a 2-manifold with corners for $j = 1, 2$. More precisely $\partial W = \partial_1 W \cup \partial_2 W \cup F_1 \cup F_2$, $\partial_j W \cap F_i = \partial_j F_i$ for $i, j = 1, 2$, $\partial_j W$ is a bordism between $\partial_j F_1$ and $\partial_j F_2$, and $\partial_1 W \cap \partial_2 W$ is a bordism from $\partial_1 F_1 \cap \partial_2 F_1$ to $\partial_1 F_2 \cap \partial_2 F_2$. The map of W restricts to the maps given by g_i on the corresponding strata.

The definition of horizontal and vertical compositions of 2-morphisms requires some cut and paste arguments but is straightforward.

There is an obvious surjection (*mor* in the category \mathcal{B})

$$mor \rightarrow H_2(imm[k], emb[k] \cup imm[k, 1]),$$

where we identify elements in $H_2(imm[k], emb[k] \cup imm[k, 1])$ with bordism classes of maps of oriented compact surfaces

$$(F, \partial F) \rightarrow (imm[k], emb[k] \cup imm[k, 2]).$$

The proof of the next result is obvious.

Proposition 1. *There exists the obvious forgetful functor*

$$\mathcal{C}(M) \rightarrow \mathcal{B}(M).$$

Obviously we can use the images of the elementary category generators of \mathcal{C} for the category \mathcal{B} . There has to be included the attaching of certain handles. We will not describe this in detail at this point, see the remark below.

Remark 8. The 2-groupoid \mathcal{B} is a *linearization* of \mathcal{C} in the following way: Let $hom[k]$ denote the set of 1-morphisms in the skein groupoid. Then $hom[k] \subset \mathbb{Z}\mathcal{L}[k+1]$ using the obvious identification. This is e. g. contained in the proof of proposition 2 in the next section. It also follows from remark 6 (ii). The additional commutation relations are of course induced by suitable maps of tori into *imm*. We have $hom[k] \neq \mathbb{Z}\mathcal{L}[k+1]$. The composition of elements in $hom[k]$ corresponds to the usual addition of homology classes. But it is only defined for classes $u, v \in H_1(emb[k] \cup imm[k, 1])$, for which $\partial(u) = z - y$, $\partial(v) = y - x$ such that $\partial(u + v) = z - x$. Then $u \circ v$ is a morphism from x to z and is represented by the homology class $u + v$. We will write $hom[k] = A(\mathcal{L}[k+1], \circ) \subset \mathbb{Z}\mathcal{L}[k+1]$ to indicate that the 1-morphisms are the subset of the free abelian group corresponding to *composable* morphisms of the 2-groupoid. Note that by the excision property of homology the explicit insertion of loops in *emb* is not necessary in the category \mathcal{B} .

In the next sections we will study the structure of the skein groupoid and its variations relevant for skein modules. We will see that all the necessary facts needed are consequences of the exact homology sequences of the pairs $imm[k], emb[k]$. More details about \mathcal{B} will be given in a different place, see [19].

In order to keep notation short we will consider the graded spaces $imm = \cup_{k \geq 0} imm[k]$. We let $\mathcal{L} := \cup_{k \geq 0} \mathcal{L}[k]$. We use the shift notation $\mathcal{L}[+1]$ and define $\mathcal{L}[+1][k] := \mathcal{L}[k+1]$.

Definition 7. Let \mathcal{R} be a commutative ring with 1. A *skein potential in \mathcal{R}* is a map

$$\sigma : \mathcal{L}[+1] \rightarrow \mathcal{R}\mathcal{L}.$$

Sometimes we will only consider

$$\sigma : \mathcal{L}[1] \rightarrow \mathcal{L}[0]$$

and call this also a skein potential.

Lemma 3. *Each skein potential defines a linear potential for the 2-category \mathcal{B} .*

Proof. This follows from the description of the 1-morphisms in \mathcal{B} . In fact, the skein potential extends to a homomorphism of abelian groups $\mathbb{Z}\mathcal{L}[+1] \rightarrow \mathcal{R}\mathcal{L}$. But $hom \subset \mathbb{Z}\mathcal{L}[+1]$ and the inclusion maps composition into sum. \square

Note that a skein potential defined $\mathcal{L}[1] \rightarrow \mathcal{R}\mathcal{L}[0]$ can easily be extended trivially to a full skein potential. So usually we will not have to distinguish between the two cases.

In sections 7 and 8 we will define interesting further deformations of the 2-groupoid $\tilde{\mathcal{B}}$. This explains the passage to Jones type skein relations and skein relations for framed oriented links.

The skein groupoid above is the result of applying a homology functor to a 2-groupoid defined by chain groups in imm (see [8] for a useful set-up of homology in this framework). The chain groups linearize the deformation 2-groupoid mentioned at the beginning of this section. It is also interesting to study skein theory on the level of chains. Then skein modules are the 0-dimensional homology modules of a chain complex with \mathcal{R} -coefficients and σ -deformed boundary operator, see [20].

4. LINK THEORY INTERPRETATION OF HOMOLOGY EXACT SEQUENCES

We consider the exact homology sequence of the pair (imm, emb) (remember the grading convention from section 3):

$$H_1(emb) \rightarrow H_1(imm) \rightarrow H_1(imm, emb) \rightarrow H_0(emb) \rightarrow H_0(imm)$$

with $i_* : H_0(emb) \rightarrow H_0(imm)$ surjective by transversality.

We want to describe the geometric meaning of the groups and homomorphisms in the portion of the exact sequence above. Obviously

$$H_0(emb) \cong \mathbb{Z}\mathcal{L},$$

Note that $H_0(imm[0]) \cong H_0(map)$, where map is the space of smooth maps in M . This is isomorphic to the free abelian group on the set of monomials in the set $\hat{\pi}$

of free homotopy classes of loops in M . So

$$H_0(\text{imm}[0]) \cong SZ\hat{\pi},$$

where S denotes the symmetric algebra. Using the above isomorphisms, the homomorphism i_* corresponds to the map \mathfrak{h} defined by assigning to each oriented singular link its homotopy class.

The description of the relative homology group is more interesting.

Proposition 2. *There is a natural isomorphism*

$$H_1(\text{imm}, \text{emb}) \cong \mathbb{Z}\mathcal{L}[+1]/\mathcal{D}$$

for subgroups $\mathcal{D}[k] \subset \mathcal{L}[k+1]$, $k \geq 0$. For $k = 0$ the subgroup $\mathcal{D}[k]$ is generated by all elements

$$K_{*+} - K_{*-} - K_{+*} + K_{-*},$$

for all $K_{**} \in \mathcal{L}[k+2]$ (differentiability relations). For $k \geq 1$ the subgroup $\mathcal{D}[k+1]$ additionally has generators corresponding to all geometric 4T-relations and tangency relations (theorem 9, (ii) and (iii)).

Proof. It follows from theorem 8 that for $k \geq 0$ the homomorphism

$$H_1(\text{imm}[k, 1] \cup \text{emb}[k], \text{emb}[k]) \rightarrow H_1(\text{imm}[k], \text{emb}[k])$$

induced by the inclusion

$$\text{imm}[k, 1] \cup \text{emb}[k] \subset \text{imm}[k]$$

is surjective. Thus we can represent each homology class by a chain, which maps into $\text{imm}[k, 1] \cup \text{emb}[k]$ with the boundary mapping into $\text{emb}[k]$. It can be assumed that the mappings of 1-simplices are transverse in the sense of section 3. Thus all parameters map into $\text{emb}[k]$, except for a finite number mapping into $\text{imm}[k, 1]$. The orientation of the 1-simplex and the usual coorientation of $\text{imm}[k, 1]$ in $\text{imm}[k]$ define a sign for each *singular* parameter. This defines an integral linear combination of elements of $\text{imm}[k, 1]$ thus an element in $H_0(\text{imm}[k, 1])$. Next consider any relative boundary. This is given by a 2-chain with boundary mapping into $\text{emb}[k]$. Now we apply lemma 2, see remark 5. Thus we can perturb each mapping of an oriented surface F into $\text{imm}[k]$ relative to the boundary such the set of parameters in F mapping into $\text{imm}[k] \setminus \text{emb}[k]$ is a 1-complex neatly embedded F with vertices of valence 4 or 2 in the interior mapping into $\text{imm}[k, 2]$. The contribution of a boundary element thus is a sum of *monodromies* around elements of $\text{imm}[k, 2]$. These elements generate the subgroup $\mathcal{D}[k+2]$ and are computed by abelianizing the relations in theorem 9. This proves the claim. \square

We summarize the discussion in the following theorem.

Theorem 11. *The homology exact sequence of the pair (imm, emb) can be identified with the following exact sequence:*

$$H_1(\text{emb}) \rightarrow H_1(\text{imm}) \xrightarrow{\mu} \mathbb{Z}\mathcal{L}[+1]/\mathcal{D} \xrightarrow{\partial} \mathbb{Z}\mathcal{L} \xrightarrow{\mathfrak{h}} \mathbb{Z}\mathfrak{b} \rightarrow 0$$

The homomorphism $\mu[k]$ is defined by the signed sum of all terms in $\mathcal{L}[k+1]$ along transversal paths in $\text{imm}[k, 1] \cup \text{emb}[k]$. The homomorphism $\partial[k]$ is the Vassiliev resolution of the last double-point

$$\mathcal{L}[k+1] \rightarrow \mathbb{Z}\mathcal{L}[k].$$

Proof. The result follows easily from proposition 1 and its proof. \square

The above sequence provides a description of the kernel of the \mathfrak{h} . This is the forgetful homomorphism from isotopy to homotopy, mapping the *quantum observables* to their semi-classical limits.

Corollary 2. *There is the isomorphism*

$$\mathbb{Z}\mathcal{L} \cong \mathbb{Z}\mathfrak{b} \oplus \text{coker}(\mu).$$

The result of the corollary just is a different description of the 2-groupoid structure discussed in section 3. It shows in a neater way the distinction in terms of *local* relations and string topology homomorphism.

Recall that we have special *geometric* splitting homomorphisms

$$\mathfrak{s} : \mathbb{Z}\mathfrak{b} \rightarrow \mathbb{Z}\mathcal{L}.$$

These are defined by realizing chord diagrams in M by corresponding immersions. Then for $k = 0$, a sequence of free homotopy classes is mapped to a link with components realizing those free homotopy classes. Recall that $\mathfrak{b}[0]$ is the set of monomials in the set $\hat{\pi}$, and

$$\mathfrak{s}[0] : \mathfrak{b} \rightarrow \mathcal{L}[0].$$

Let $\mathfrak{s}[0](\alpha) =: K_\alpha \in \mathcal{L}[0]$ be the standard link corresponding to the monomial $\alpha \in \mathfrak{b}$.

In the following let σ denote either a skein potential or the Vassiliev potential σ_V .

Proof of theorems 1 and 2 (Conway boundary case): All claims follow already from theorems 5-8 and theorem 10 applied to the categories $\tilde{\mathcal{B}}[0]$ and $\tilde{\mathcal{B}}$, see the discussion at the end of section 3. The models are $\mathfrak{b}[0]$ respectively \mathfrak{b} . The exact sequence of theorem 8 (or the splitting in corollary 2) describes in a systematic way *all* homology classes of 1-cycles with boundary $K - \mathfrak{s} \circ \mathfrak{h}(K)$ for $K \in \mathcal{L}$. The additional information we get is that the relations coming from expansion of *closed* 1-morphism up to 2-morphisms, *factor* through the homomorphism μ . In the inductive argument we actually have to consider the lift of $\tilde{\sigma} \circ \mu$ to $\mathcal{R}\mathcal{L}$. Here we have defined:

$$\tilde{\sigma} : \mathbb{Z}\mathcal{L}[+1]/\mathcal{D} \rightarrow \mathcal{R}\mathcal{L}/\langle \sigma(\mathcal{D}) \rangle,$$

where $\langle \sigma(\mathcal{D}) \rangle$ is the submodule generated by the subgroup $\sigma(\mathcal{D})$. □

Remark 9. Consider the skein module case Let $\iota : \mathcal{R}\mathcal{L}[0] \rightarrow \mathcal{S}(\sigma)$ be the projection. Then $\iota \circ \tilde{\sigma} \circ \mu = 0$. Here we use that ι factors through $\mathcal{R}/\langle \mathcal{D} \rangle$ and in $\mathcal{S}(\sigma)$ the following relation holds:

$$\sigma(K_{*+} - K_{*-} - K_{+*} - K_{-*}) = \partial(K_{*+} - K_{*-} - K_{+*} - K_{-*}) = K_{++} - K_{--} \dots$$

But for $K_+ \neq K_-$ a skein relation of the form $K_+ - K_- - \sigma(K_*)$ is not contained in the image of $\tilde{\sigma} \circ \mu$, even if we consider the extended homomorphism

$$\tilde{\sigma} \circ \mu : H_1(\text{imm}) \otimes \mathcal{R} \rightarrow \mathcal{R}\mathcal{L}/\langle \sigma(\mathcal{D}) \rangle.$$

In fact, there is a more natural way of associating a string topology homomorphism to a given skein relation or Vassiliev relation. This will be discussed in section 9.

5. GENERAL RESULTS ABOUT SKEIN POTENTIALS AND SKEIN MODULES

We work in the graded 2-groupoid $\tilde{\mathcal{B}}$ or in the 2-groupoid $\tilde{\mathcal{B}}[0]$. Then $\text{hom}[k]$ is naturally identified with $\mathbb{Z}\mathcal{L}[k+1]$. Because of the functoriality property in the definition a skein potential is determined by its values on the generators $\mathcal{L}[k+1]$. Thus we consider a skein potential as a map $\mathcal{L}[+1] \rightarrow \mathcal{L}$ using the usual graded notation.

For some of the results we have to restrict the choice of geometric models. Let

$$\sigma : \mathcal{L}[1] \rightarrow \mathcal{RL}[0]$$

be a skein potential with respect to the ideal $I \subset \mathcal{R}$. Let $K_\alpha := \mathfrak{s}(\alpha)$ be the *standard link* with homotopy class α .

Definition 8. A choice of geometric models

$$\mathfrak{s} : \mathfrak{b}[0] \rightarrow \mathcal{L}[0]$$

is called *nice* if the following two conditions hold: (i) for free homotopy classes which are multiple times contained in a monomial $\alpha \in \mathfrak{b}[0]$ there exists a self-isotopy of the standard link K_α , which arbitrarily changes the order of the corresponding components. (ii) for each trivial free homotopy class in α , the standard link K_α contains some unlinked and unknotted circles in a 3-ball separated from the rest of the link.

Lemma 4. *It is always possible to choose nice geometric models.*

Proof. If α contains a free homotopy class $\beta \in \hat{\pi}$ with multiplicity n_β then we can choose n_β parallel copies in the link K_α . Of course trivial homotopy classes can be represented by unlinked and unknotted components separated from the link. \square

Remark 10. Let M be the solid torus $S^1 \times D^2$, or a Lens space $L(p, q)$. Then $L(p, q)$ contains a solid torus and each link is isotopic into this torus. Then we can choose nice standard links contained in $S^1 \times D^2$, which are descending. In fact, there is an isotopy of the solid torus which changes the order of components with multiple free homotopy classes of components.

Definition 9. A skein potential $\sigma : \mathcal{L}[+1] \rightarrow \mathcal{RL}$ is called *local* if it is of the form

$$\sigma(K_*) = \sum_{i=1}^n h_i K_i,$$

where for $1 \leq i \leq n$, $a_i \in \mathcal{R}$ and K_i are singular links defined by replacing the two intersecting arcs in the oriented 3-ball centered about the last double-point $*$ by the 2-tangles t_i . We assume that $a_i \in I$ if the number of components of K_i is not smaller than the number of components of K_* .

The following is obvious.

Lemma 5. *Each local skein potential*

$$\sigma : \mathcal{L}[1] \rightarrow \mathcal{L}[0]$$

extends to a local skein potential

$$\mathcal{L}[+1] \rightarrow \mathcal{L}.$$

In fact for each $1 \leq i \leq k+1$ there exist maps

$$\sigma_i : \mathcal{L}[k+1] \rightarrow \mathcal{L}[k]$$

defined by applying σ to the i -th double-point of a singular link. \square

Similarly we can define Jones and Conway boundaries on $\mathcal{L}[k+1]$ for $1 \leq i \leq k+1$ by applying ∂ to the i -th double point. Note that for $i < j$

$$\partial_i \circ \sigma_j = \sigma_{j-1} \circ \partial_i.$$

Recall that a skein relation is insensitive (with respect to differentiability relations for $k \geq 1$ if for each differentiability element $d \in \mathcal{L}[k+1]$ we can choose a preimage $\delta^{-1}(\sigma(d))$ such that $\sigma(\delta^{-1}(\sigma(d))) = 0$.

Remark 11. Global insensitivity for the skein potentials of the graded category is rare because of the $4T$ -relations. In fact, the usual Vassiliev potential σ_V is *not* robust. Also the extensions of skein relations according to lemma 4 are not robust. In particular the following result only applies to the usual skein potentials.

Proposition 3. *Each local skein potential $\mathcal{L}[1] \rightarrow \mathcal{RL}[0]$ is robust. Also, each local skein potential $\mathcal{L}[+1] \rightarrow \mathcal{RL}$ is robust with respect to the differentiability morphisms.*

Proof. Each differentiability element d is in the image of

$$\partial_{k+2} - \partial_{k+1} : \mathcal{L}[k+2] \rightarrow \mathbb{Z}\mathcal{L}[k],$$

which maps $K_{**} \in \mathcal{L}[k+2]$ to

$$K_{*+} - K_{*-} - K_{+*} - K_{-*}.$$

We have to show that for each $y \in \mathcal{L}[k+2]$ we can choose a preimage, denoted $\partial^{-1}(y)$, such that

$$\sigma \circ \partial^{-1} \circ \sigma \circ \partial_{k+2}(y) = \sigma \circ \partial^{-1} \circ \sigma \circ \partial_{k+1}(y).$$

Recall that σ without index operates on the last double-point. Of course we can choose the preimage on the left hand side of the equation such that $\partial^{-1} \circ \sigma \circ \partial_{k+2}(y) = \sigma_{k+1}(y)$ and similarly on the right hand side $\partial^{-1} \circ \sigma \circ \partial_{k+1}(y) = \sigma(y)$. This follows from locality, e. g. in the first case: $\sigma \circ \partial_{k+2} = \partial \circ \sigma_{k+1}$. Then the claim follows using locality:

$$\sigma \circ \sigma_{k+1} = \sigma \circ \sigma_{k+2}.$$

\square

In the following it will sometimes turn out to be useful to describe skein modules $\mathcal{S}(\sigma)$ as modules over the corresponding skein module $\mathcal{S}(D^3)$.

Proposition 4. *For each local skein potential σ the module $\mathcal{S}(M; \sigma)$ is a module over the commutative skein algebra $\mathcal{S}(D^3; \sigma)$, and a module over the skein algebra $\mathcal{S}(\partial M; \sigma)$.*

Proof. The disjoint union with links in a 3-ball separated from a given link is a well-defined operation. Locality of the skein potential implies that this defines a pairing

$$\mathcal{S}(D^3; \sigma) \otimes \mathcal{S}(M; \sigma) \rightarrow \mathcal{S}(M; \sigma).$$

The same holds for $\mathcal{S}(\partial M; \sigma)$ using a collar of ∂M . In fact, the case of D^3 is a special case where we replace M by the complement of some open 3-ball. Because

of locality this does not change the skein module. Also this allows to identify the skein modules of D^3 and $S^2 \times [0, 1]$. The algebra structures on $\mathcal{S}(M; \sigma)$ is defined in the usual way using the $[0, 1]$ -structure. \square

If ∂M is not a union of tori then the skein algebra usually is not commutative.

Let σ be the *Conway skein potential* $\sigma(K_*) = hK_0$ respectively zK_0 for a double-point of the same respectively different components. In this case we add the vacuum relation $(q^{-1} - q)\emptyset = hU$. Note that $hU \neq \sigma(K_*)$ for any singular link $K_* \in \mathcal{L}[1]$. Recall that by definition our category contains the empty link and thus the empty 1-morphism in $\mathcal{L}[1]$, which is the identity morphism of the empty link. When working with vacuum relations we actually only replace the trivial skein relation $\sigma(\emptyset) = (q^{-1} - q)\emptyset$ e. g. in the Jones case (such that $(\partial - \sigma)(\emptyset) = 0$ automatically holds) by the relation above. This will have the advantage that the kink relations (see section 6) can be already included into the structure of the ring $\mathfrak{R} := \mathcal{S}(D^3; \sigma)$.

6. THE HOMOMORPHISM μ AND THE TOPOLOGY OF M

For $k \geq 1$ let $\mathcal{K}[k] \subset \mathcal{L}[k]$ denote the set of k -immersions with a distinguished self-crossing denoted $*$ such that one of the lobes of the singular component of K_* bounds a disk intersecting K_* only in its boundary along the lobe. We call this a *kink k -immersion*. There is a unique homology class $\gamma(K_*) \in H_1(\text{imm}[k-1])$ represented by the transversal path in $\text{imm}[k-1]$, which is defined by running through the natural isotopy from K_+ to K_- and the crossing change at K_* . The disk bounding the trivial component in the smoothing K_0 defines a null-homology of this loop in $\text{map}[k-1]$. This null-homology contains a single point contained in $\text{map}[k-1] \setminus \text{imm}[k-1]$, where the corresponding map has a circle which embeds except at a single point with vanishing tangent vector. The corresponding elements is non-trivial in $H_1(\text{imm}[k-1])$. Note that

$$(\mu[k-1])(\gamma(K_*)) = K_*,$$

and

$$(\partial[k-1])(K_*) = K_+ - K_- = 0$$

by construction for each $K_* \in \mathcal{K}[k]$.

Remark 12. Let $\tilde{\mathcal{L}}[k]$ denote the set of isotopy classes of ordered singular k -links (the set of path components of the space $\widetilde{\text{emb}}[k]$ of the space of ordered k -embeddings). Then there is a well-defined *onto* map

$$\chi[k] : \tilde{\mathcal{L}}[k-1] \rightarrow \mathcal{K}[k],$$

which assigns to each ordered $k-1$ -singular link the immersion with some additional kink in the *first* component away from possible preimages of double-points.

Lemma 6. *The kernel of the epimorphism*

$$j_* : H_1(\text{imm}) \rightarrow H_1(\text{map})$$

is generated by all elements of the form $\gamma(K_)$ for all $K_* \in \mathcal{K}[+1]$. In particular the linear extension of χ maps $\mathbb{Z}\tilde{\mathcal{L}}$ onto $\ker(j_*)$.*

Proof. This is another application of Lin transversality, respectively a modification of lemma 1 and remark 5 in section 3. Consider a mapping of a surface F into $map[k]$, which is transversal along ∂F . It can be approximated relative ∂F to the following way: The set of points in F , which map into $map[k] \setminus emb[k]$ consists of a 1-complex embedded in F , with vertices of possible orders 2, 4 or 1 in the interior. Those of order 4 or 2 are mapped to $immk[k, 2]$. Those of order 1 are mapped to a smooth map, which is an immersion with k double points and a single point in the complement of the double points with vanishing tangent. The point with vanishing tangent appears in the boundary of $imm[k, 1] \subset imm[k]$. Now cut out small disks from F around the vertices of order 1. The restriction to the the boundary of a disk represents an element in $H_1(imm[k])$ of the form $\gamma(K_*)$ for $K_* \in \mathcal{K}[k]$. \square

Let in the presentation $K_{**} \in \mathcal{K}[2]$ the first place indicate a self-crossing with a bounding lobe. Then K_{*+}, K_{*-} are contained in $\mathcal{K}[1]$. This is not necessarily the case for K_{+*}, K_{-*} but $K_{+*} = K_{-*}$. Thus we have the following commuting diagram with exact rows, where \mathcal{D}_\bullet is the subgroup of $\mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1])$, which is generated by the differentiability relations with *all* four terms in $\mathcal{L}[1] \setminus \mathcal{K}[1]$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D} \cap \mathbb{Z}\mathcal{K}[1] & \longrightarrow & \mathcal{D}[0] & \longrightarrow & \mathcal{D}_\bullet & \longrightarrow & 0 \\ & & \subset \downarrow & & \subset \downarrow & & \subset \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}\mathcal{K}[1] & \longrightarrow & \mathbb{Z}\mathcal{L}[1] & \longrightarrow & \mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1]) & \longrightarrow & 0 \end{array}$$

and the diagram of homomorphisms

$$\begin{array}{ccc} H_1(imm[0]) & \xrightarrow{\mu[0]} & \mathbb{Z}\mathcal{L}[1]/\mathcal{D}[0] \\ i_*[0] \downarrow & & \downarrow \\ H_1(map[0]) & \xrightarrow{\mu_\bullet} & \mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1])/\mathcal{D}_\bullet \end{array}$$

with the vertical right map defined by projection.

Corollary 3. *There is the induced boundary homomorphism ∂_\bullet and the exact sequence*

$$H_1(emb[0]) \rightarrow H_1(map[0]) \xrightarrow{\mu_\bullet} \mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1])/\mathcal{D}_\bullet \xrightarrow{\partial_\bullet} \mathbb{Z}\mathcal{L}[0] \xrightarrow{\mathfrak{h}} SZ\hat{\pi}.$$

There is the induced isomorphism

$$coker(\mu[0]) \cong coker(\mu_\bullet).$$

Proof. Most of the claims follow by the construction of the homomorphisms. The exactness in $H_1(imm[0])$ follows because $\mu[0]$ restricts to an epimorphism

$$\mathfrak{k}[0] \rightarrow \mathbb{Z}\mathcal{K}[1]/(\mathcal{D}[1] \cap \mathbb{Z}\mathcal{K}[1]).$$

\square

The main link theoretic consequences will be based on the result of theorem 2.

Definition 10. Let M be a compact 3-manifold. M is called *atoroidal* (respectively *aspherical*) if each π_1 -injective map of a torus (respectively map of a 2-sphere) in M is homotopic into ∂M

Each irreducible 3-manifold is aspherical. Note that each hyperbolic 3-manifold is aspherical and atoroidal.

Theorem 12. *Suppose M is aspherical and atoroidal. Then $\mu_\bullet = 0$.*

Proof. Most of the arguments are already contained in [16] and [17]. Fix a component map_α of the space map corresponding to $\alpha \in \mathfrak{b}$. Note that

$$H_*(map) \cong \bigoplus_{\alpha \in \mathfrak{b}} H_*(map_\alpha).$$

The Hurewicz theorem

$$\pi_1(map, f_\alpha) \rightarrow H_1(map_\alpha)$$

is onto. Let \widetilde{map} be the space of ordered smooth maps with the fat diagonal excluded. Let \widetilde{map}_α denote the preimage of map_α under the covering projection

$$\widetilde{map} \rightarrow map.$$

Note that the path components of \widetilde{map}_α are labelled by the orderings of α . Let a be an ordered sequence of elements in $\hat{\pi}$ corresponding to α . We choose a representative embedding $\widetilde{f}_a \in \widetilde{map}_\alpha$. Note that f_a is ordered. If a does not contain multiple elements of $\hat{\pi}$ then the components \widetilde{map}_α is homeomorphic to map_α . But multiple occurrences of elements of $\hat{\pi}$ in a imply that there are homotopies joining ordered links with the same underlying unordered link. Then the covering of the component is non-trivial and the injection

$$\pi_1(\widetilde{map}_\alpha, f_a) \rightarrow \pi_1(map, f_\alpha)$$

is not necessarily onto. This means that a loop in imm_α does not necessarily lift to a loop in imm_a . In this case we choose the embeddings f_a in the following symmetric way: If some element in a appears multiple times then we choose corresponding parallel components for f_a . Thus there exists an isotopy of M which changes the order of components. We can compose a given loop γ in map_α with loops in emb changing the order in such a way that the composition of loops lifts to a loop in map_a . Note that the composition still has the same image under μ_\bullet as γ . So the image of μ_\bullet on the component $H_1(map_\alpha)$ corresponds to the image of a homomorphism defined on

$$\pi_1(map; f_a) \cong \prod_i \pi_1(map_{a_i}, f_{a_i}).$$

Note that we can fill in the infinite codimensional fat diagonal without changing the fundamental group and then identify our mapping space with the product of single component mapping spaces. This shows that the image of μ_\bullet is generated by the images of elements represented by loops in imm which fix all but one component. Let L denote the union of those components which are fixed during the homotopy. In the following note that free homotopy implies homology.

Next consider the singular torus map given by the adjoint of the non-constant component of such a loop in imm . First assume that this map is *not* essential. Then the image of $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$ is cyclic (for details see [16]). Thus on a neighbourhood of wedge $S^1 \vee S^1$ in $S^1 \times S^1$ the map can be homotoped into the tubular neighbourhood T of a knot embedded in M . The full torus map is homotopic in T up to a map of a 2-sphere. We can easily arrange that the singular torus can be represented as a connected sum of singular torus in N and parallel copies of 2-spheres contained in the collar neighbourhood of the union of 2-sphere

components in ∂M . Now consider a singular torus contained in T . We want to apply the homotopy exact sequence of the fibration (see [46], Appendix):

$$\Omega M \rightarrow \text{map} \rightarrow M,$$

where Ω is the usual based loop functor for $M = T$ a solid torus. It is easy to see that all elements of $\pi_1(\text{map}; f)$ can be represented by loops in $\text{emb}[0](1)$. (Each generator of $\pi_1(\text{map}; f)$ can be represented by longitudinal rotation of the knot f in T .) The possible connected sum arcs for the 2-sphere contribution can be homotoped along. But since the two-spheres and connecting tubes can be assumed embedded the resulting loop is still contained in $\text{emb}[0](1)$. Finally since the core of T and connecting arcs are 1-dimensional and the 2-sphere maps into a collar of the boundary, the intersections with L can be avoided and the loop of maps is contained in emb . Thus the image under μ_\bullet vanishes.

It remains to discuss the case of an essential torus map. If this map is homotopic into the boundary then the boundary contains a torus and we can homotope into a collar neighbourhood $N \cong S^1 \times S^1 \times [0, 1]$ of a torus boundary component, in particular avoiding L . As before we can argue in the mapping space of N and find a free homotopy of the torus map thus inducing a homology of the given loop into emb . \square

Conjecture 1. If $\mu_\bullet = 0$ then M is aspherical and atoroidal.

The conjecture is easy to show except for a few small Seifert fibred spaces. We will discuss this problem in [21].

Example 3. (a) Let $M = S^3$ and consider a *connected sum* element in $\mathcal{L}[1]$. Thus $*$ is a self-crossing and K_0 is the union of two nonempty links contained in disjoint balls. Then $K_+ = K_-$ by rotation. The full 2π -rotation defines a loop ℓ in imm . Note that $\mu(\ell)$ is represented by K_* so obviously not trivial in $\mathcal{L}[1]$. Now consider a path in imm , which joins one of the two link pieces in K_0 with trivial link by only crossing changes. Since the rotation can be performed along this deformation there exists a free homotopy of the loop ℓ into a loop for which the rotation is a kink rotation. Thus this phenomenon is not measured by μ_\bullet .

(b) Suppose M is not aspherical. Then either M contains $S^2 \times S^1$ or M is a connected sum. In each case it is easy to show that $\mu_0 \neq 0$, at least modulo the Poincare conjecture. For example in the second case take a connected sum immersion with each lobe homotopically non-trivial. A natural isotopy across the S^2 easily changes K_+ into K_- . But now K_* cannot be trivial in $(\mathbb{Z}\mathcal{L}[1] \setminus \mathcal{K}[1])/\mathcal{D}_\bullet$. In order to show this map into $S\mathbb{Z}\hat{\pi}$ by taking homotopy classes.

7. THE JONES DEFORMATION

In this section we describe the passage from the Conway boundary to the Jones boundary case. There is a natural way to introduce the q -structure abstractly on the groupoid level. In the topological case this amounts to a lift of twisted homology to the deformed fundamental groupoid.

Let \mathcal{C} be a 2-groupoid and let

$$\varepsilon : \text{hom}/\text{mor} \rightarrow \mathbb{Z}$$

be a *twist* homomorphism (or abstract local system), i. e.

$$\varepsilon(u \circ v) = \varepsilon(u) + \varepsilon(v)$$

for all $u, v \in \text{hom}$. (We identify elements in hom with its equivalence classes under the action of mor .) It follows that

$$\varepsilon(u^{-1}) = -\varepsilon(u)$$

since $\text{mor}(u \circ u^{-1}, 1) \neq \emptyset$ and $\varepsilon(1) = 0$ for each identity 1-morphism $1 \in \text{hom}(x, x)$.

We describe a 2-groupoid \mathcal{C}_q , the *Jones deformation* of \mathcal{C} . To avoid confusion we will write the compositions in the category \mathcal{C}_q as \diamond .

Let for $i = 0, 1, 2$

$$(\text{hom}_i)_q := \{q^j w \mid w \in \text{hom}_i, j \in \mathbb{Z}\},$$

where we use the notation $q^j w := (j, w)$, i. e. $(\text{hom}_i)_q$ is the subset of scalar multiples of basis elements in free abelian group on hom_i . To avoid confusion we write $q^0 u$ for the image of $u \in \text{hom}_i$ in $(\text{hom}_i)_q$ for all i .

There is the natural action of \mathbb{Z} on $(\text{hom}_i)_q$ defined by $q^k w := q^{j+k} u \in (\text{hom}_i)_q$ for $w = q^j u$, $u \in \text{hom}_i$ and $j, k \in \mathbb{Z}$ and $i = 0, 1, 2$.

We define the target and source maps and the compositions. If $u \in \text{hom}(x, y)$ then let $q^0 u \in \text{hom}_q(x, q^{-2\varepsilon(u)} y)$ and $q^i u \in \text{hom}_q(q^i x, q^{i-2\varepsilon(u)} y)$ for $i \in \mathbb{Z}$. Thus

$$\text{source}_q(q^i u) = q^i \text{source}(u) \in \mathfrak{ob}_q$$

and

$$\text{target}_q(q^i u) = q^{i-2\varepsilon(u)} \text{target}(u) \in \mathfrak{ob}_q$$

for each $u \in \text{hom}$. The source and target maps are *equivariant* with respect to the \mathbb{Z} -action.

Let $v \in \text{hom}(x, y)$ and $u \in \text{hom}(y, z)$. It follows from the *homomorphism* property of ε that, if $u \circ v$ is defined then

$$q^{-2\varepsilon(v)} u \diamond q^0 v := q^0(u \circ v)$$

is defined, and is a 1-morphism from x to $q^{-2\varepsilon(u)-2\varepsilon(v)} y = q^{-2\varepsilon(u \circ v)} y$ in \mathcal{C}_q . A composition with $q^i v$ is defined such that equivariance holds:

$$q^i(a \diamond b) = (q^i a) \diamond (q^i b)$$

for $a, b \in \text{hom}_q$. Note that $\text{hom}_q(q^i x, q^j y) = \emptyset$ if $j - i$ is odd. Thus \mathfrak{ob}_q naturally splits into a disjoint union of two sets with homomorphisms only between objects in each of the two sets.

Finally we define horizontal and vertical composition of 2-morphisms. Note that if $\text{mor}(u, v) \neq \emptyset$ then $\varepsilon(u) = \varepsilon(v)$. Thus after identification of $u, v \in \text{hom}$ with the corresponding elements in hom_q (note that the targets have changed) we can identify each $h \in \text{mor}(u, v)$ with the corresponding 2-morphism in mor_q . We will have $q^i h \in \text{mor}_q(q^i u, q^i v)$ for $h \in \text{mor}(u, v)$. The composition in mor induces obvious compositions \diamond_1, \diamond_2 in mor_q .

Theorem 13. (i) $\mathcal{C}_q = (\mathfrak{ob}_q, \text{hom}_q, \text{mor}_q)$ defines a 2-groupoid.

(ii) There is an obvious functor of 2-groupoids $p_q : \mathcal{C}_q \rightarrow \mathcal{C}$ defined by $q \mapsto 1$, which is onto on all sets hom_j for $j = 0, 1, 2$.

(iii) There is a natural ε -twisted functor $i_q : \mathcal{C} \rightarrow \mathcal{C}_q$, where $i_q(a) := q^0 a$ for $a \in \text{hom}_i$. This means that for $u, v \in \text{hom}$:

$$i_q(u \circ v) = q^{-2\varepsilon(v)} u \diamond q^0(v).$$

For $v_1, v_2 \in \text{hom}(x, y)$, $u_1, u_2 \in \text{hom}(y, z)$ and $g \in \text{mor}(u_1, u_2)$, $h \in \text{mor}(v_1, v_2)$ we have

$$i_q(g \circ_2 h) = (q^{-2\varepsilon(v_1)}g) \diamond_2 q^0 h.$$

For $h_1 \in \text{mor}(v_2, v_3)$ and $h_i \in \text{mor}(v_1, v_2)$ with $v_i \in \text{hom}(x, y)$ for $i = 1, 2, 3$ we have the untwisted identity

$$i_q(h_1 \circ_1 h_2) = (q^0 h_1) \diamond_1 (q^0 h_2).$$

(iv) The equation of functors holds: $p_q \circ i_q = \text{id}$.

Proof. Let $x, y \in \mathbf{ob}$. and $u \in \text{hom}(x, y)$ with uniquely determined $u^{-1} \in \text{hom}(y, x)$. Then $(q^0 u)^{-1} := q^{-2\varepsilon(u)}u^{-1} \in \text{hom}_q(q^{-2\varepsilon(u)}y, x)$. We are using that $\varepsilon(u^{-1}) = -\varepsilon(u)$. The inverse in general is defined by equivariance

$$(q^i u)^{-1} := q^i (q^0 u)^{-1}.$$

for $i \in \mathbb{Z}$. The rest of the arguments are straightforward and left to the reader. \square

Remark 13. In terms of graphs (see [4] for the graphical explanation of categories) this means that we take the original graph corresponding to the category \mathcal{C} and take $\mathbb{Z} \times \mathbf{ob}$. Then define the edges according to the above procedure determined by the twist. This is like lifting paths into a covering space corresponding to ε . Note that the twist is a local system on the graph associated to the 2-groupoid.

Theorem 14. Let $\mathcal{R} \supset \mathbb{Z}$ be a commutative ring with 1. Let \mathcal{C} be a 2-groupoid with linear potential $\mathbf{a} : \text{hom} \rightarrow \mathcal{R}\mathbf{ob}$. Suppose that $\text{hom} = F(S, \circ)$ or $\text{hom} = A(S, \circ)$ for a subset $S \subset \text{hom}$. There exists the skein potential in $\mathcal{R}[q^{\pm 1}]$:

$$\mathbf{a}_q : \text{hom}_q \rightarrow \mathcal{R}\mathbf{ob}_q \cong \mathcal{R}[q^{\pm 1}]\mathbf{ob},$$

defined uniquely by functoriality and \mathbb{Z} -equivariance:

$$\mathbf{a}_q(q^0 s) := \mathbf{a}(s)$$

for all $s \in S$

By assumption the following equation holds:

$$\mathbf{a}_q((q^0 s)^{-1}) = \mathbf{a}(s^{-1}) = -q^n \mathbf{a}(s).$$

The next useful formula is easily proved from the definitions.

Proposition 5. Let $\mathcal{C}, \mathbf{a}, S$ be as in the last theorem. Let $s_j \in S$ and $\epsilon_j \in \{\pm 1\}$ for $j = 1, \dots, r$. Then the following formula holds:

$$\mathbf{a}_q(i_q(s_r^{\epsilon_r} \circ s_{r-1}^{\epsilon_{r-1}} \circ \dots \circ s_1^{\epsilon_1})) = \sum_{j=1}^r \epsilon_j q^{-\epsilon_j \varepsilon(s_j)} \mathbf{a}(s_j)$$

Remark 14. Consider the Vassiliev groupoid $\mathcal{C}(M)$ respectively the skein groupoid $\mathcal{B}(M)$. Then $\text{hom}[k] = F(\mathcal{L}[k+1], \circ)$ respectively $\text{hom}[k] = A(\mathcal{L}[k+1], \circ)$. We define $\varepsilon(K^\pm) = \pm 1$ for $K \in \mathcal{L}[k+1]$. Note that K_* corresponds to the morphism s_* defined by the crossing change from $-$ to $+$ at $*$. We have $\text{hom}_q[k] = \text{hom}[k]_k \subset \mathbb{Z}[q^\pm \mathcal{L}[k+1]]$ in a similar way (see also the description of $\mathbb{Z}[q^\pm \mathcal{L}[k+1]]$ as twisted homology below). The following is important. In the q -deformed case K_* corresponds to the morphism defined by $K_- \mapsto q^{-2}K_+$ and K_*^{-1} corresponds to the morphism $q^{-2}K_+ \mapsto K_-$, which the q^{-2} -multiple of the usual morphism $K_+ \mapsto q^2K_-$. It is this correspondence which makes $\text{hom}_q \subset \mathbb{Z}[q^\pm \mathcal{L}[+1]]$ a morphism.

Let σ be a skein potential in \mathcal{R} . By theorem 14 σ induces a skein potential $\mathcal{L}[+1] \rightarrow \mathcal{RL}$ in the following way. Note that σ induces a linear potential in the usual way. This linear potential induces a linear potential in the deformed category \mathcal{B}_q . Using the identification from remark 14 this defines a skein potential

$$\sigma_q : \mathcal{L}[+1] \rightarrow \mathcal{R}[q^\pm]\mathcal{L}.$$

Then

$$\sigma_q(i_q(s_r^{\epsilon_r} \circ s_{r-1}^{\epsilon_{r-1}} \circ \dots \circ s_1^{\epsilon_1})) = \sum_{j=1}^r \epsilon_j q^{-\epsilon_j} \sigma(s_j)$$

where $s_j = K_{*,j}$ is a composable sequence of elements in $\mathcal{L}[k+1]$.

Definition 11. We say that $(\mathcal{C}, \varepsilon)$ has *kinks* if for each $x \in \mathfrak{ob}$ there exists a kink morphism $k_x \in \text{hom}(x, x)$ such that $\varepsilon(k_x) = 1$.

Now suppose $(\mathcal{C}, \varepsilon)$ has kinks and let \mathfrak{m} be a set of models of \mathcal{C} . Then obviously $\mathfrak{m}_q := \mathfrak{m} \cup q\mathfrak{m}$ is a set of models of the category \mathcal{C}_q . Let $i \in \mathbb{Z}$. For $x \in \mathfrak{ob}$ we can find $u \in \text{hom}(x, m)$. By possibly iterated composition with kink morphisms we can arrange that $\varepsilon(u) \in \{-i, -i-1\}$. Thus $q^i u \in \text{hom}(q^i x, m)$ or $q^i u \in \text{hom}(q^i x, qm)$. Now assume that $I \subset R$ is an ideal and \mathfrak{c} is a model complexity on \mathfrak{m} . Then we define \mathfrak{c}_q by $\mathfrak{c}_q(m) = \mathfrak{c}_q(qm) = \mathfrak{c}(m)$. Let $I_q \subset \mathcal{R}[q^\pm]$ be the ideal generated by I in the extension. Then it is easy to see that if \mathfrak{a} is linear with respect to I and \mathfrak{c} then \mathfrak{a}_q is linear with respect to I_q and \mathfrak{c}_q .

Thus we can apply theorem 8 to $(\mathcal{C}_q, \mathfrak{a}_q, \mathfrak{m}_q)$. and deduce the map

$$\mathfrak{ob}_q \rightarrow (\mathcal{R}[q^\pm]\mathfrak{m}_q)[[I_q]]/\mathcal{I}_q,$$

with the submodule \mathcal{I}_q defined in the obvious way.

In the topological case, which we consider here, the passage from $q = 1$ to Jones theory is more easily described by introducing a *local system* on *imm*. This viewpoint has already been used in [K2]. We refer there and to [W] for technical details. These system appear naturally in trying to work the kink contributions into the theory as will be seen in the following.

Throughout this section we assume $k = 0$ and omit the index.

Let $R := \mathbb{Z}[q^\pm]$. The idea is that a *universal* Jones type relation should be of the form:

$$q^{-1}K_+ - qK_- - K_*$$

respectively

$$q^{2\epsilon}K_\epsilon - K_{-\epsilon} - \epsilon K_* = 0$$

for $\epsilon = \pm 1$.

Consider the trivial bundle over *imm* with fiber R . Let γ be a loop in *imm* which is tansversal (compare e. g. theorem 1). Then the *oriented intersection number with the discriminant* $\delta'(\gamma)$ is well-defined. Formally it can be defined as the homomorphism:

$$\delta' : H_1(\text{imm}) \xrightarrow{\mu} \mathbb{Z}\mathcal{L}[1]/\mathcal{D} \longrightarrow \mathbb{Z},$$

where the second homomorphism is defined my mapping each element of $\mathbb{Z}\mathcal{L}[1]$ to the generator $1 \in \mathbb{Z}$. For fixed basepoints f_α in the components of *imm* (possibly

represented by $\mathfrak{s}(\mathfrak{b})$) let

$$\pi_1(\text{imm}) := \bigcup_{\alpha \in \mathfrak{b}} \pi_1(\text{imm}; f_\alpha)$$

Then define the map, which restricts to homomorphisms on the fundamental groups of the path components of imm :

$$\delta : \pi_1(\text{imm}) \rightarrow \text{Aut}(\mathbb{Z}[q^{\pm 1}])$$

is defined by mapping a loop γ to the multiplication by $q^{-2(\delta' \circ H)(\gamma)}$. Here we use the Hurewicz map

$$H : \pi_1(\text{imm}) \rightarrow H_1(\text{imm}).$$

The map δ induces the local system \tilde{R} on imm . It is called the *Jones local system* since it is related to the Jones respectively Homfly polynomial skein relation.

Remark 15. It is possible to describe the local system *directly* as a functor on the *transversal fundamental groupoid* of imm . The objects of this category are points in emb and the morphisms are homotopy classes of paths in imm joining two embeddings. Now represent a path in imm with endpoints in emb relative to the boundary by a transversal path. Then compute the oriented intersection number with the discriminant. So in this case we apply the homomorphism

$$H_1(\text{imm}, \text{emb}) \rightarrow \mathbb{Z}\mathcal{L}[1]/\mathcal{D} \rightarrow \mathbb{Z}$$

with the first homomorphism defined in the proof of proposition 1. Now the boundary operator on chains with coefficients in R is easily defined as usual using the *parallel transport* on suitable paths. Thus our construction actually given by a transversal chain construction as usual in string topology.

Note that the local system is trivial over emb . Also by using the kink loops described in section 2 we see that

$$H_0(\text{imm}; \tilde{R}) \cong SR\hat{\pi}/(q^2 - 1) \cong SZ\hat{\pi} \oplus qSZ\hat{\pi}.$$

Let $\tilde{\mathcal{D}}$ be the submodule of $R\mathcal{L}[1]$ generated by all elements of the form

$$q^{-1}K_{*+} - qK_{*-} - q^{-1}K_{+*} + qK_{-*}$$

and let

$$\tilde{\partial} : R\mathcal{L}[1]/\tilde{\mathcal{D}} \rightarrow R\mathcal{L}$$

be defined by

$$\tilde{\partial}(K_*) = q^{-2}K_+ - K_-.$$

Note that

$$\tilde{\partial}(q^{-1}K_{*+} - qK_{*-} - q^{-1}K_{+*} + qK_{-*}) = 0.$$

The following theorem is proved in the same way as theorem 1.

Theorem 15. *The homology exact sequence for the local system \tilde{R} on the pair of spaces (imm, emb) is isomorphic to the exact sequence:*

$$H_1(\text{emb}; R) \rightarrow H_1(\text{imm}; \tilde{R}) \xrightarrow{\tilde{\mu}} R\mathcal{L}[1]/\tilde{\mathcal{D}} \xrightarrow{\tilde{\partial}} R\mathcal{L} \xrightarrow{\tilde{\mathfrak{h}}} SR\hat{\pi}/(q^2 - 1)$$

with $\tilde{\mathfrak{h}}$ onto and the homomorphism $\tilde{\mu}$ described below.

The homomorphism $\tilde{\mu}$ has the following description: Consider a 1-chain representing a homology class in $H_1(\text{imm}; \tilde{R})$. We can approximate the maps on 1-simplices by transversal maps, i. e. the boundaries mapping into emb . Now let a simplex s be given by a transversal map

$$\beta : ([0, 1], \{0, 1\}) \rightarrow (\text{imm}, \text{emb})$$

and let $0 < t_1 < t_2 < \dots < t_n < 1$ be the parameters which map into $\text{imm}[1]$. Let the sign at t_i be $\epsilon_i \in \{0, 1\}$, with $\epsilon_i = -1$ if the path crosses the discriminant from the *positive to the negative side*.

$$\tilde{\mu}(s) = \sum_{i=1}^n \epsilon_i q^{2(\epsilon_1 + \dots + \epsilon_{i-1}) + \epsilon_i} K_{*,i},$$

where $K_{*,i}$ is the isotopy class of $\beta(t_i)$.

Remark 16. Combining 1-simplices with cancelling boundary terms in a 1-chain it follows easily that there is an epimorphism

$$R\pi_1^0(\text{imm}) \rightarrow H_1(\text{imm}; \tilde{R}),$$

where $\pi_1^0(\text{imm})$ is the union (over \mathfrak{b}) of the subgroups of homotopy classes of those loops with trivial index (thus the 1-simplex representing the loop has trivial boundary with local coefficients). Note that the effect of changing the basepoint of a loop in $f \in \text{emb}$ representing an element in $\pi_1^0(\text{imm}; f)$ to a basepoint g by pre- and postcomposition with a path from g to f and the reverse path multiplies by a power of q . Thus we can restrict to a single basepoint in each component.

From the exact sequence in theorem we conclude:

Corollary 4. *There is the short exact sequence of R -modules:*

$$0 \rightarrow (R\mathcal{L}[1]/\tilde{\mathcal{D}}) / \text{im}(\tilde{\mu}) \rightarrow R\mathcal{L} \rightarrow SR\hat{\pi}/(q^2 - 1) \rightarrow 0,$$

This sequence is *never* split in the category of R -modules (but in the category of abelian groups). Note that, for a given $K \in \mathcal{L}$, there is a unique $\alpha \in \mathfrak{b}$ such that for each integer number n

$$K - q^{2n} K_\alpha \in \ker(\tilde{\mathfrak{h}}).$$

In physics terms this can be interpreted as follows: *For each classical state we can choose a quantum state up to a certain phase.*

By application of the homotopy Conway map (defined in section 4) the description of the homomorphism $\tilde{\mu}$ given above precisely corresponds to the map on paths in [K1].

Proposition 6. *Suppose that M is aspherical and atoroidal. Then*

$$\text{im}(\tilde{\mu}) \subset (RK/(\mathcal{D} \cap RK)) \cap \ker(\tilde{\mathfrak{h}})$$

More precisely, each element in the image of $\tilde{\mu}$ is represented by a sum of elements of the form

$$q^{n_i} (K_{*,i} - K'_{*,i}),$$

where for fixed i all $K_{,i}, K'_{*,i}$ are immersions in \mathcal{K} resulting from the same link $K \in \mathcal{L}$ by introducing a kink in one of its components.*

Proof. First define homomorphisms for each $\alpha \in \mathfrak{b}$

$$\tilde{\mu}_\alpha : \pi_1(\text{imm}; f_\alpha) \rightarrow R\mathcal{L}[1]/\tilde{\mathcal{D}}$$

using the explicit formula given above. Note that the resulting homomorphism does not factor through a map defined on the set of free homotopy classes of loops in imm . Now assume we have given a loop ρ in imm with index $\varepsilon(\rho) = 0$. The geometric arguments in the proof of theorem 2 provide a homotopy to a product

$$\prod_{1 \leq i \leq n} \ell_i \rho_i \ell_i^{-1}.$$

Here ℓ_i is a transverse arcs joining $\rho(*) \in \text{emb}$ (representing a link $K \in \mathcal{L}$) to the basepoint $*_i$ on ρ_i , and each ρ_i is a kink loop (representing some $\gamma(K_*)$ for some K_* in $\mathfrak{L}[1]_{\mathfrak{t}}$). Now each path $\ell_i \rho_i \ell_i^{-1}$ is homotopic inside imm to a kink path for the basepoint embedding. Note that the order of crossing changes can be altered up to $\tilde{\mathcal{D}}$. This allows to inductively change each of the paths. Thus finally we have a composition of kink loops for the embedding $*$ with vanishing index because the index is homotopy invariant. This proves the claim. Note that the kinks can still be contained in different components of K . \square

8. THE FRAMING LOCAL SYSTEMS AND ITS HOMOLOGY SEQUENCES

In this section we use a local system of coefficients R' on imm naturally related to the study of framed links in M . The geometric interpretation of the corresponding homology exact sequence will be much more involved.

The following setup has been discussed in [K2]. A *total framing* of an immersion $f \in \widetilde{\text{imm}}$ is a choice of equivalence class $[v]$ of a normal vectorfield of f (section of the normal bundle), where we will have $v_1 \equiv v_2$ if v_1 is homotopic to v_2 , or if v_1 differs from v_2 by twisting the framings of components of f such that the total number of those twists add up to zero. Obviously the set of total framings is in 1 – 1 correspondence with the integer numbers. The notation of total framing easily extends to unordered immersions by dividing by the actions of symmetric groups in the obvious way. Now for each $f \in \text{imm}$ let R'_f denote the free abelian group generated by the total framings on f . Then $R'_f \cong \mathbb{Z}[q^{\pm 1}] = R$ with q^i and q^j indicating two framings of f , which differ by $j - i$ twists.

Then a local system $R_{\mathfrak{f}}$ on (imm, emb) with bundle of groups given by

$$\bigcup_{f \in \text{imm}} R_f$$

is defined from the collection of homomorphisms, for $f \in \text{emb}$:

$$\delta_{\mathfrak{f}} : \pi_1(\text{imm}; f) \rightarrow \text{Aut}(R_f)$$

by assigning to a loop γ in f the multiplication by $q^{\delta'_{\mathfrak{f}}(\gamma)}$. Here $\delta'_{\mathfrak{f}}(\gamma) \in \mathbb{Z}$ is the sum of $2(\varepsilon \circ H)(\gamma)$ and the framing change induced by the loop. It is shown in [K2] using Lin transversality that the homomorphism is well-defined and thus represents a local system $R_{\mathfrak{f}}$ on imm . Moreover, the homology module $H_0(\text{imm}; R_{\mathfrak{f}})$ is isomorphic to the skein module $\mathcal{S}_{\mathfrak{f}}$ defined by dividing $R\mathcal{L}_{\mathfrak{f}}$ by the submodule generated by all elements $q^{-1}K_+ - qK_-$ and $q^{-1}K^{(+)} - K$. Here $\mathcal{L}_{\mathfrak{f}}$ is the usual set of isotopy classes of framed oriented links and for each framed oriented link K the framed

link $K^{(+)}$ is the framed link defined from K by introducing a positive twist into the framing of any of its components. The obviously defined set of isotopy classes of totally framed immersions with k self-intersections in M will be denoted $\mathcal{L}[k]_{\mathfrak{t}}$. In particular $\mathcal{L}_{\mathfrak{t}}$ is the set of isotopy classes of totally framed links in M . Note that the local system restricts to the subsets $imm[k] \subset imm$.

Remark 17. The local system above (and similarly the one in section 3) admits an interpretation in terms of the following general construction: Let $\tilde{X} \rightarrow X$ be a (not necessarily regular) covering space over a connected space X . Let F be the fiber over the basepoint $* \in X$ and let $\pi_1(X; *) \rightarrow \text{homeo}(F)$ be the associated monodromy map. Then there is the natural map

$$\text{homeo}(F) \rightarrow \text{Aut}(\mathbb{Z}F),$$

where $\text{homeo}(F)$ is the group of homeomorphisms of F . So we can form the associated covering space with fiber $\mathbb{Z}F$ and define the *induced* monodromy by composition in the obvious way. Now assume that F is a group acting on itself by translation. Thus we have a natural map $F \rightarrow \text{homeo}(F)$. Assume that, using a stratification of X , there is defined a second monodromy map:

$$\pi_1(X; *) \rightarrow F \rightarrow \text{homeo}(F).$$

The two monodromies into $\text{homeo}(F)$ can be multiplied, and then mapped to a homomorphism into $\text{Aut}(\mathbb{Z}F)$ as above. The fiber of the induced covering is the abelian group $\mathbb{Z}F$ and the monodromy defines a local system, see [W]. In section 3 the trivial covering has been used.

Lemma 7. *For k any non-negative integer:*

$$H_0(imm[k]; R_{\mathfrak{f}}) \cong R\mathcal{L}[k]_{\mathfrak{t}}/(q^{-1}K^{(+)} - K) \cong R\mathcal{L}[k]_{\mathfrak{f}}/(q^{-1}K^{(+)} - K) \cong \mathbb{Z}\mathcal{L}_{\mathfrak{t}}.$$

The right hand isomorphism is an isomorphism of abelian groups.

Proof. First consider the case $k = 0$. The fiber R_f over a given embedding f is the free abelian group generated by total framings of f . R_f is identified with R by a choice of total framing of f . Multiplication by q corresponds to a positive twist. Note that

$$H_0(emb; R_{\mathfrak{f}}) \cong \bigoplus_{K \in \mathcal{L}} H_0(emb_K, R_{\mathfrak{f}}),$$

where emb_K is the set of embeddings in the isotopy class K . By definition emb_K is the $\text{Diff}(M, id)$ -orbit of a representative embedding f . Here $\text{Diff}(M, id)$ is the group of diffeomorphisms of M , which are isotopic to the identity. Then each element of $H_0(emb_K; R_{\mathfrak{f}})$ is represented by an element in the fiber over some representative embedding of the isotopy class. Now for $k = 0$ the result follows from the definitions of $\delta_{\mathfrak{f}}$ and $\mathcal{L}_{\mathfrak{t}}$ and the monodromy interpretation of 0-dimensional homology for local systems [W]. Consider an integer number $k \geq 1$. There are defined maps over compact sets in $imm[k]$, which apply to all double-points the positive resolution. Then use that $\text{Diff}(M, id)$ acts on all sets $imm[k]$, and the action is compatible with the resolution maps and framings. Note that by definition also two elements in $imm[k]$ are isotopic if and only if they differ by the action of $\text{Diff}(M, id)$. \square

We will use the notation

$$\mathcal{F}[k] := R\mathcal{L}[k]_{\mathfrak{t}}/(q^{-1}K^{(+)} - K)$$

for all k , and as usual $\mathcal{F}[0] =: \mathcal{F}$.

Theorem 16. *The homology exact sequence for the local system $R_{\mathfrak{f}}$ on the pair (imm, emb) is isomorphic with the exact sequence:*

$$\begin{array}{ccccccc} H_1(emb; R_{\mathfrak{f}}) & \rightarrow & H_1(imm; R_{\mathfrak{f}}) & \xrightarrow{\mu_{\mathfrak{f}}} & \mathcal{F}[1]/\mathcal{D}_{\mathfrak{t}} & & \\ & & \xrightarrow{\partial_{\mathfrak{f}}} & \mathcal{F} & \xrightarrow{\mathfrak{h}_{\mathfrak{f}}} & \mathcal{S}_{\mathfrak{f}} & \rightarrow 0 \end{array}$$

The submodule $\mathcal{D}_{\mathfrak{t}}$, the homomorphisms $\mu_{\mathfrak{f}}$, $\partial_{\mathfrak{f}}$ and $\mathfrak{h}_{\mathfrak{f}}$ are defined as before by replacing \mathcal{L} with $\mathcal{L}_{\mathfrak{t}}$. In the case of $\mu_{\mathfrak{f}}$ one has to consider the transportation of framings.

Proof. First use the same arguments as before to get the exact sequence

$$\begin{array}{ccccccc} H_1(emb; R_{\mathfrak{f}}) & \rightarrow & H_1(imm; R_{\mathfrak{f}}) & \rightarrow & H_0(imm[1]; R_{\mathfrak{f}})/\partial'(H_0(imm[2]; R_{\mathfrak{f}})) & & \\ & & & & \rightarrow & H_0(emb; R_{\mathfrak{f}}) & \rightarrow \mathcal{S}_{\mathfrak{f}} \rightarrow 0 \end{array}$$

and apply the lemma. Note that $\mathcal{S}_{\mathfrak{f}}$ is isomorphic to the quotient of $R\mathcal{L}_{\mathfrak{t}}$ by the submodules generated by elements $q^{-1}K^{(+)} - K$ and $q^{-1}K_+ - qK_-$ (compare [K2]). \square

Remark 18. It is interesting to note that the exact sequence above maps into the sequence of theorem 1 by using the coefficient homomorphism $R \rightarrow \mathbb{Z}$, which maps q to 1. Then the coefficient system becomes trivial and $\mathcal{F}[k]$ maps onto $\mathbb{Z}\mathcal{L}[k]$. In this way theorem 4 appears as a *framing quantization* of theorem 1.

Corollary 5. *There is the short exact sequence of R -modules*

$$0 \rightarrow (\mathcal{F}[1]/\mathcal{D}_{\mathfrak{t}})/im(\mu_{\mathfrak{f}}) \rightarrow \mathcal{F} \rightarrow \mathcal{S}_{\mathfrak{f}} \rightarrow 0$$

It is known that the homomorphism $\mathfrak{h}_{\mathfrak{f}}$ splits in the category of R -modules if and only if each mapping $S^1 \times S^1 \rightarrow M$ is homotopic into ∂M .

Theorem 17. (i) *There is the isomorphism of R -modules*

$$\mathcal{F}[k] \cong \bigoplus_{\alpha \in \mathfrak{b}} R(\mathfrak{h}[k]^{-1}(\alpha))/(q^{2\varepsilon_0(\alpha)} - 1),$$

where $\mathfrak{h}[k] : \mathcal{L}[k] \rightarrow \mathfrak{b}$ is the map taking homotopy classes of components, for k any non-negative integer. The index $\varepsilon_0(\alpha) \in \mathbb{N}$ is defined by the the gcd over all absolute values of total oriented intersection number with 2-spheres in M .

(ii) *There is the isomorphism of R -modules*

$$\mathcal{S}_{\mathfrak{f}} \cong \bigoplus_{\alpha \in \mathfrak{b}} R/(q^{2\varepsilon(\alpha)} - 1),$$

where $\varepsilon(\alpha)$ is the gcd of absolute values of intersection numbers of singular tori (defined by sweeping a component α_i through M) with a link realizing α .

Proof. Assertion [i] follows from Chernov's results [C] and the authors results in [K2]. This extends to the case $k \geq 1$ using the argument in the proof of the lemma above. \square

Remark 19. (i) By defining suitable torus maps from 2-spheres as in (i) it can be proved that $\varepsilon_0(\alpha)$ is a multiple of $\varepsilon(\alpha)$ (see [K2]). The homomorphism $\mathfrak{h}_{\mathfrak{f}}$ thus is defined by composition of \mathfrak{h} with some obvious projection. (Use that $x^{nm} - 1 = (x^n - 1)(1 + x^n + x^{2n} + \dots + x^{(n-1)n})$.)

(ii)

Corollary 6. (i) *Suppose that M does not contain any non-separating 2-spheres. Then $\mathcal{F}[k] \cong R\mathcal{L}[k]$, for all nonnegative integer numbers k .*

(ii) *Suppose that each mapping from $S^1 \times S^1$ into M is homologous into ∂M . $\mathcal{S}_f \cong SR\hat{\pi}$.* \square

Proof. Consider $k = 0$. By choosing total framings we can define a section $\psi : \mathcal{L} \rightarrow \mathcal{L}_t$ of the forget map $\phi : \mathcal{L}_t \rightarrow \mathcal{L}$. The induced homomorphism $\chi : R\mathcal{L} \rightarrow R\mathcal{L}_t$ is obviously onto. It follows from Chernov's [C] results that the fibers of ϕ are in 1-correspondence with \mathbb{Z} . This implies that χ is injective. (If $\chi(q^n K - K) = 0$ then $\psi(K)$ and its n -fold twist would be isotopic.) \square

Remark 20. The proof of corollary 5 shows that the sections ψ and \mathfrak{s} have analogous meanings in the theory. Let $K \in \mathcal{L}_t$ with $\mathfrak{h}(\phi(K)) = \alpha$. Then

$$\mathfrak{h}_f(K - q^j \psi(K_\alpha)) = 0 \in \mathcal{S}_f,$$

where the sections ψ and \mathfrak{s} used to define the isomorphism of theorem 5(ii), if and only if by applications of the skein relations to K we can get

$$\psi(K_\alpha) \bmod (q^{2\varepsilon(\alpha)} - 1).$$

Corollary 7. *Suppose that each map of a torus into M is homologous into ∂M . Then the homology exact sequence for the local system R_f on (imm, emb) is isomorphic to the exact sequence:*

$$\begin{array}{ccccccc} H_1(emb; R) & \rightarrow & H_1(imm; R) & \xrightarrow{\mu_f} & R\mathcal{L}[1]/\tilde{\mathcal{D}} & & \\ & & \xrightarrow{\partial_f} & R\mathcal{L} & \xrightarrow{\mathfrak{h}_f} & SR\hat{\pi} & \rightarrow 0 \end{array}$$

\square

Proof. The assumption implies that the monodromy homomorphism δ_f is trivial. So the homology for the local system is the usual homology with coefficients in R . \square

Remark 21. The homomorphism \mathfrak{h}_f composed with the projection homomorphism $SR\hat{\pi} \rightarrow SR\hat{\pi}/(q^2 - 1)$ is the homomorphism $\tilde{\mathfrak{h}}$ in theorem 3. Note that μ_f is defined on $H_1(imm; R) \cong H_1(imm) \otimes R$ while $\tilde{\mu}$ is defined on the homology with local coefficients defined by \tilde{R} .

Proposition 7. *Suppose that M is apherical and atoroidal. Then $\mu_f = 0$.*

Proof. It follows from the assumption that in particular each torus is homologous into the boundary. Thus we can assume we are in the situation of the exact sequence of corollary 5. Since $H_1(imm; R) \cong H_1(imm) \otimes R$ it suffices to show that μ_f vanishes on the image of the natural homomorphism of abelian groups $H_1(imm) \rightarrow H_1(imm) \otimes R$. That μ_f vanishes on \mathfrak{k} is obvious by definition of the local system. (In fact the kink isotopy induces a change of framing cancelling the contribution of the crossing.) Thus we actually consider a homomorphism

$$H_1(map) \rightarrow \mathcal{F}[1]/\mathcal{D}_f \cong R\mathcal{L}[1]/\tilde{\mathcal{D}}.$$

The rest of the argument follows word by word the geometric reasoning in the proof of theorem 2. \square

9. COMPUTATION OF SKEIN MODULES

Rees algebra [12], 6.5 noetherian is because image of noetherian, note that \mathfrak{R} is a subring of domain, see [3], chapter 5 for localization (polynomials to laurent polynomials)

for general results about completions see [3], chapter 10

The fact: \mathfrak{R} is a noetherian domain implies that the inclusion of a free module into the completion is injective by the corollary of Krull's theorem. Notice that the module is free but not finitely generated. [3], p. 110

10. SOME RELATIONS WITH STRING TOPOLOGY

In order to relate the homomorphisms μ respectively μ_0 with string topology operations we have to deal with both a passage from isotopy to homotopy, and a multiplication (respectively transversely a smoothing) operation. In turns out to be interesting to describe this in the two possible ways of applying these operations in different order.

The ad hoc arguments used in the proof of theorem 2 hint at difficulties in a passage from ordered to unordered maps. The Chas Sullivan construction is a construction in the homology of *ordered* maps of circles into M .

We first recall basic features of their setup and restrict our viewpoint at this moment to the two fundamental string topology operations in the case of 3-manifolds. Let $\widetilde{top}(j)$ denote the space of continuous (or piecewise smooth) mappings $\cup_j S^1 \rightarrow M$. Moreover let $\widetilde{top}(j)_o$ denote the subspace of those maps with at least one constant component. The group $(S^1)^j$ acts on the space $\widetilde{top}(j)$ preserving the subspace $\widetilde{top}(j)_o$. This can be used to define equivariant homology groups $H_*^{eq}(\widetilde{top}(j))$ and relative equivariant homology groups $H_*^{eq}(\widetilde{top}(j), \widetilde{top}(j)_o)$.

Following [K1] and [K2] the two basic string operations c and s are the collision and self-collision operators. If restricted to 1-dimensional homology they are homomorphisms

$$c : H_1^{eq}(\widetilde{top}(2)) \rightarrow H_0(\widetilde{top}(1))$$

and

$$s : H_1^{eq}(\widetilde{top}(1)) \rightarrow H_0(\widetilde{top}(2), \widetilde{top}(2)_o)$$

Remark 22. In the case of rational coefficients we have the Künneth isomorphisms

$$H_*^{eq}(\widetilde{top}(j), \widetilde{top}(j)_o) \cong \bigotimes_j H_*^{eq}(\widetilde{top}(1), \widetilde{top}(1)_o).$$

In [CS2] the operations c and s are actually described geometrically in terms of $H_*^{eq}(\widetilde{top}(j), \widetilde{top}(j)_o)$. But Chas and Sullivan prefer to use the Künneth identification to express the operations in terms of $H_*(\widetilde{top}(1), \widetilde{top}(1)_o)$ denoted \mathbb{L} in [CS2]. We have already identified the equivariant and nonequivariant 0-dimensional homology groups.

It follows easily from the definitions that operations, also denoted s, c here, can be defined as follows

$$s : H_1^{eq}(\widetilde{imm}(1)) \rightarrow H_0(\widetilde{imm}(2)).$$

and similarly

$$c : H_1^{eq}(\widetilde{imm}(2)) \rightarrow H_0(\widetilde{imm}(1))$$

Here we also used that the inclusion from continuous maps into immersions is an isomorphism in 0 dimensional homology.

The Chas Sullivan construction allows to introduce *dummy* components, which are insensitive to collisions or self-collisions. An example would be

$$s_i : H_1^{eq}(\widetilde{imm}(j)) \rightarrow H_0(\widetilde{imm}(j+1)),$$

which measures self-collisions in the i -th component, $1 \leq i \leq j$. In the same way there are defined for $j \geq 2$:

$$c_{k\ell} : H_1^{eq}(\widetilde{imm}(j)) \rightarrow H_0(\widetilde{imm}(j-1)),$$

measuring the collisions between the k -th and ℓ -th component for $1 \leq k < \ell \leq j$. By taking the sum of the s_i respectively $c_{k\ell}$ we get well-defined homomorphisms with domain and target as above. Finally by summation over all j we have defined homomorphisms:

$$s, c : H_1^{eq}(\widetilde{imm}) \rightarrow H_0(\widetilde{imm})$$

and thus a homomorphism (s, t) . This can be precomposed with the surjective homomorphism

$$H_1(\widetilde{imm}) \rightarrow H_1^{eq}(\widetilde{imm})$$

to define

$$\tilde{j} : H_1(\widetilde{imm}) \rightarrow H_0(imm) \oplus H_0(imm),$$

where we have in the domain we have composed with the mapping induced by the projection $\widetilde{imm} \rightarrow imm$. (The surjectivity is proved as follows: First decompose

$$H_1^{eq}(\widetilde{imm}) \cong \bigoplus_{a \in \tilde{\mathfrak{b}}} H_1^{eq}(\widetilde{imm}_a),$$

where the $\tilde{\mathfrak{b}}$ is the set of ordered sequences in $\hat{\pi}$. Then apply the Künneth theorem to decompose for a of length n :

$$H_1^{eq}(\widetilde{imm}_a) \cong \bigotimes_{1 \leq i \leq n} H_1(imm_{a_i}/S^1).$$

Consider the Gysin epimorphisms (see [K2] and [C1])

$$H_1(imm_{a_i}) \rightarrow H_1(imm_{a_i}/S^1).$$

Finally apply Künneth isomorphism to non-equivariant homology to deduce the result.)

Consider the decomposition

$$H_1(\widetilde{imm}) \bigoplus_{a \in \tilde{\mathfrak{b}}} H_1(imm_a).$$

and the Hurewicz epimorphisms for $a \in \mathfrak{b}$:

$$\pi_1(\widetilde{imm}; f_a) \rightarrow H_1(\widetilde{imm}_a).$$

The definition of c and s immediatly imply that \tilde{j} factors through the image of $\pi_1(\widetilde{imm}_a)$ in $\pi_1(imm_a)$, where $a \in \mathfrak{b}$ is the unordered sequence underlying a . Finally the argument in the proof of theorem 2 shows that this homomorphism

naturally extends to homomorphisms on the full groups $\pi_1(\text{imm}_\alpha)$. Because commutators map trivially it factors through the homology groups $H_1(\text{imm}_\alpha)$. Using linearity and summarizing we thus have defined the homomorphism:

$$j : H_1(\text{imm}) \rightarrow H_0(\text{imm}) \oplus H_0(\text{imm}).$$

We want to consider j with the homomorphism μ . We first show that j results from μ by applying a smoothing construction followed by the passage from isotopy to homotopy.

Let

$$c : \mathbb{Z}\mathcal{L}[1] \rightarrow \mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L}$$

be defined by mapping basis elements K_* to the Conway smoothing K_0 , placed into the first summand for a self-crossing and into the second summand for a crossing of distinct components of K_* .

For $j \geq 0$ let $\mathcal{L}[k](j)$ denote the subset of $\mathcal{L}[k]$ given by immersions with j components. Then more precisely the homomorphism defined above is graded by homomorphisms:

$$c(j) : \mathbb{Z}\mathcal{L}[1](j) \rightarrow \mathbb{Z}\mathcal{L}(j+1) \oplus \mathbb{Z}\mathcal{L}(j-1)$$

for $j \geq 1$. Of course c induces the homomorphism

$$\mathbb{Z}\mathcal{L}[1]/\mathcal{D} \rightarrow \mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L}/c(\mathcal{D}).$$

It follows that there is the well-defined homomorphism

$$c \circ \mu : H_1(\text{imm}) \rightarrow (\mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L})/c(\mathcal{D}).$$

assigning to each transversal loop in imm the oriented sum of Conway smoothings at singular parameters. Now note that

$$\mathfrak{h} \oplus \mathfrak{h} : \mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L} \rightarrow \mathbb{Z}\mathfrak{b} \oplus \mathbb{Z}\mathfrak{b}$$

maps $\sigma(\mathcal{D})$ to 0. Thus we have defined the homomorphism

$$(\mathfrak{h} \oplus \mathfrak{h}) \circ c \circ \mu : H_1(\text{imm}) \rightarrow \mathbb{Z}\mathfrak{b} \oplus \mathbb{Z}\mathfrak{b} \cong H_0(\text{imm}) \oplus H_0(\text{imm}).$$

The following result is now obvious from the definitions.

Theorem 18.

$$(\mathfrak{h} \oplus \mathfrak{h}) \circ c \circ \mu = j : H_1(\text{imm}) \rightarrow H_0(\text{imm}) \oplus H_0(\text{imm})$$

11. APPENDIX

In [45] Vassiliev describes the approach of finite dimensional models for the mapping spaces we have considered in this paper.

Thom-ismorphism theorem for homology with twisted coefficients, [42], p. 283.
tangent bundle of $\text{imm}(1)$ [7], p. 111

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