TUBE-EQUIVALENCE OF SPANNING SURFACES AND SEIFERT SURFACES

by

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The thesis presented by *Thomas Glass* entitled *Tube-Equivalence of Spanning Surfaces and Seifert Surfaces* is hereby approved.

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ABSTRACT

Utilizing the tools familiar to the knot theorist, i.e., the Reidemeister moves, the Seifert algorithm, and cut and paste, Bar-Natan, Fulman, and Kauffman have proved that spanning surfaces are tube-equivalent for possibly disconnected spanning surfaces. In this paper, connectivity is added to the assumption and we show: *If* S_1 *and* S_2 *are Seifert surfaces for the link* L*, then* S_1 *and* S_2 *are tube-equivalent.* The proof proceeds by examining how changes to a projection of a link affect the corresponding Seifert surfaces. Maintaining connectedness of a surface allows for controlling the first homology and the Seifert pairing by S-equivalenc, and thus, is used in proving that the Alexander polynomial of the given link is an invariant.

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CHAPTER 1

Introduction

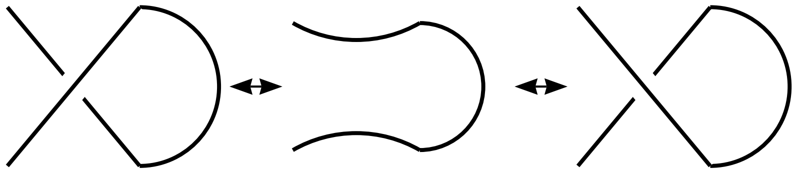
1.1 The Knot Toolbox.

The two main proofs in this paper utilize: the Reidemeister (R) moves, and the Seifert algorithm. In this section we will describe these two notions. A knot is a smooth (or piecewise linear) embedding of S^1 in $\mathbb{R}^3$. A link is a smooth embedding of disjoint copies of S^1 into $\mathbb{R}^3$. The embedding provides an orientation for the link. Two links L_1 and L_2 of L are (ambient) isotopic if there exists a homeomorphism of the ambient space $\mathbb{R}^3$ that maps L_1 to L_2. Any two projections of isotopic links differ from each other by a sequence of R moves (see Sec 1.1.1) and planar isotopy. For each projection there is defined a spanning surface. But a given link can have many different spanning surfaces. By different surfaces, we mean that the surface embeddings are not even ambient isotopic. For example, if surfaces have different genera, then they are clearly not isotopic [6]. In general it is a difficult problem to decide whether two surfaces are isotopic.

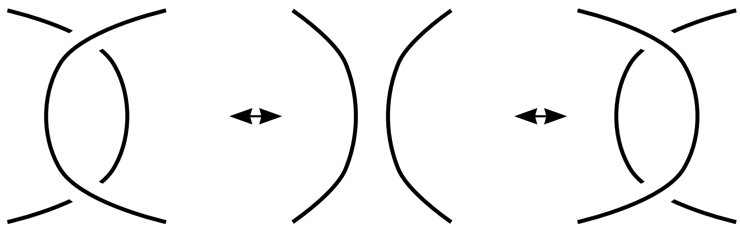
R moves allow one to study ambient-isotopy of links. The Seifert algorithm associates to each projection an oriented spanning surface. The R moves and the Seifert algorithm will be used to analyze the projections of L and the associated spanning surfaces.

1.1.1 The Reidemeister (R) moves.

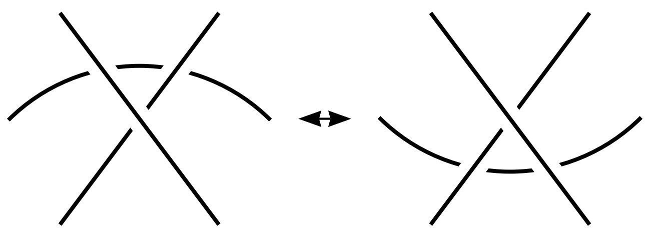
R1)



R2)



R3)



1.1.2 The Seifert Algorithm.

Given a knot or link L, Seifert first proved existence and provided a means of constructing an orientable surface with boundary L from any planar projection of L. Begin with a projection of a link. Then, starting at an arbitrary point on an arc, trace around the diagram in the direction of the orientation. Any time a crossing is met, change arcs along which you trace, but do so in such a way that the tracing continues in the direction of the knot. If at some point you start retracing your path, go to an untraced portion on the diagram and begin tracing again. Because there are only finitely many arcs the procedure will stop eventually. [3] The resulting diagram is a disjoint union of circles in the plane without crossings, called *Seifert circles*.

Seifert circles may be nested, one inside another. For a Seifert circle C, define the *depth* of C, denoted $d(C)$, to be the number of Seifert circles in which it is properly contained. Translate each circle C upward (toward the reader) by $d(C)$ units, and attach a disk inside each C so that each Seifert circle bounds a disk. Assign colors to disks in order to track the orientation of the surface. Let disks with clockwise orientation be colored gray, and disks with counterclockwise orientation be colored white. One can imagine a piece of cloth that is gray on one side and white on the other.

Where there existed crossings between the Seifert circles, now attach bands with a half twist that matches where the previous crossing was removed. Any band that connects disks of the same height will have a half twist to preserve orientation. The other set of bands will connect disks that differ in height by one unit. The disks will have the same color showing. A band with a half twist brings the color of the upper side of the lower disk to match the upper disk [7]. The result is an oriented surface with boundary L a spanning surface for the link.

Figure 1.1 illustrates the result of this procedure applied to the figure eight.

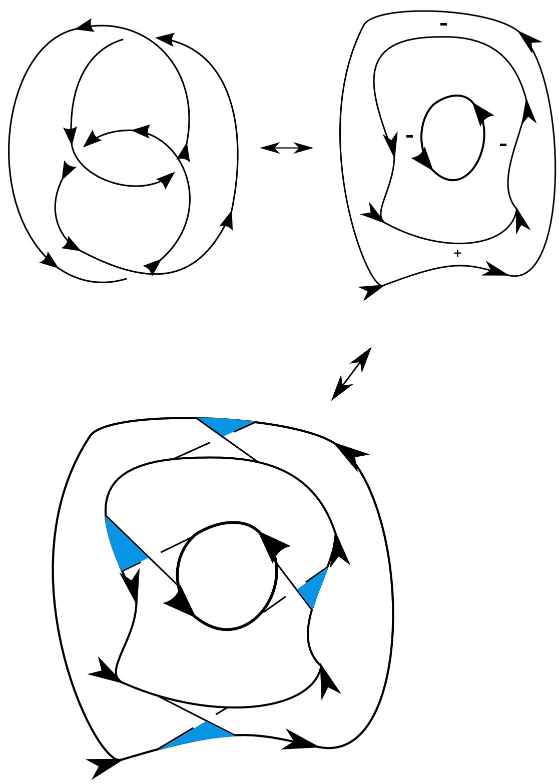


Figure 1.1: A Seifert suface of the figure 8

CHAPTER 2

Tube-Equivalence of Spanning Surfaces.

Definition 1a. A *spanning surface* S of a link L is a compact, 2-dimensional manifold with oriented boundary in S^3 with no closed components such that the S has oriented boundary L.

Definition 1b. A *Seifert surface* is a connected spanning surface.

Definition 2. Two spanning surfaces of a link L are *tube-equivalent* if they are related by isotopy and the attaching or deleting of fillable tubes.

A tube T is fillable if it extends to an embedding including its co-core disk D_T such that D_T does not intersect the surface except in ∂D_T, see Figure 2.1

Theorem 1. *Let S_1 and S_2 be two spanning surfaces for the same link L. Then S_1 and S_2 are tube-equivalent.*

This theorem, and the later Theorem 2, can can be proved using elementary Morse theory and the Thom-Pontrjagin construction [9]. However, it can also be proved using standard tools of knot theory, i.e., Seifert Surfaces and Reidemeister moves.

We are concerned with spanning surfaces of links. There are times when compressing a tube or an R move may disconnect a spanning surface of a link, but the

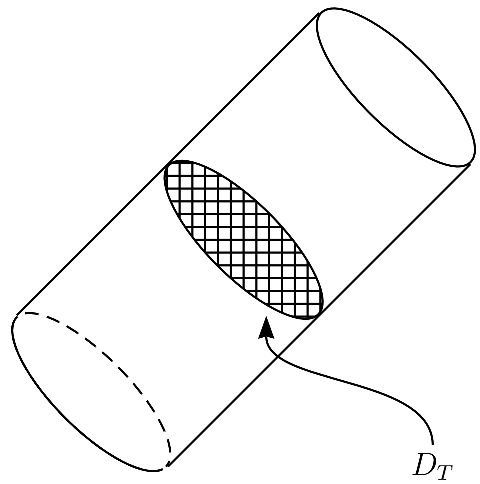


Figure 2.1: Fillable tube.

surface is still a spanning surface of L. Theorem 1 is not concerned that surfaces remain connected; however, this is necessary to define the Alexander polynomial from the surface. In Theorem 2 connectivity will be added to the assumption. Following is a reworking of the recent paper by Bar-Natan, Fulman and Kauffman [8].

Let "$\uparrow$" indicate a Seifert circle and T denote a tube. Combinations of four symbols will be placed between these lines. Let "+" denote a positive crossing and "−" denote a negative crossing. Seifert circles are connected with twisted bands. Let the bands be black or gray on one side and white or stripped on the other side, depending on the diagram. This coloring is important to insure surface orientation. A crossing denoted by "?" can be either a positive or negative crossing. For the rest of this proof let "=" indicate isotopy and "∼" for tube-equivalence. Positive and negative crossings are shown in Figure 2.2 (note that the orientation of L is used here in an essential way).

Proving Theorem 1 proceeds by proving four Lemmas. Each lemma will utilize

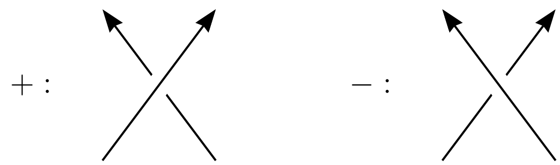


Figure 2.2: Two types of oriented crossings.

constructing a spanning surface for L using the Seifert algorithm. Seifert's algorithm always produces a surface obtained by connecting a set of disks by twisted bands, resulting in a surface with no closed components. For a connected spanning surface we can always choose a sequence of the connecting bands to untwist and widen until the set of disks becomes a single disk with a set of bands attached. If the surface is not connected we still have a disk with bands attached for each component of the surface. This new form is called band-handle form, and, in fact, it is easy to show that any spanning surface without closed components is ambient isotopic to a surface in band-handle form [4] [5]. The figure-eight from Figure 1.1 in band-handle form is shown in Figure 2.3.

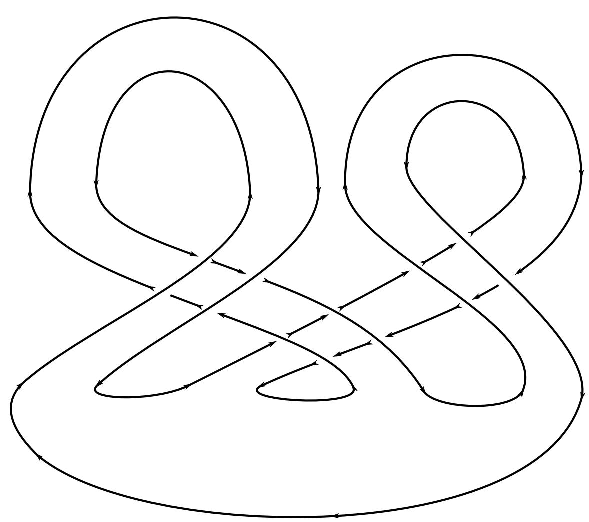


Figure 2.3: Band-handle form of the figure-eight

Before proceeding, a word of clarification is necessary. The reason that we define that a spanning surface has no closed components is that a closed surface cannot be isotoped to band-handle form. But we can avoid this problem as follows: Suppose that a surface S bounding a link Lhas closed components. Let F be the union of its closed components. Then first connect all the closed components by tubes and call the surface comprised out of the closed components F'. Attach F' to $S \setminus F$ by a tube and call the resulting surface S'. Then by Mayer-Vietoris sequence arguments, $H_1(S) \cong H_1(S')$ [1]. Thus we can always disregard closed components and only consider the components of S with boundary. This is why we chose to define spanning surfaces in Definition 1a without closed components.

2.0.3 A First Tubing Relation.

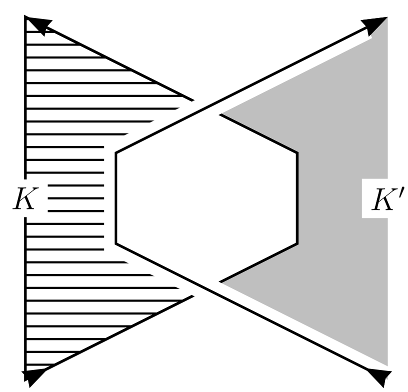


Figure 2.4: Two Seifert circles connected by bands with opposite twists

Lemma 1. $\uparrow \pm \uparrow = \uparrow T \uparrow = \uparrow \mp \uparrow$. *Here "=" denotes isotopy of embedded surfaces. Connecting two surfaces with two adjacent oppositely twisted bands with opposite twists is equivalent or isotopic to adding a tube between them.*

Proof. We give the proof pictorially. Consider the surface corresponding to $\uparrow \pm \uparrow$.

This translates into Seifert circles joined by two twisted bands, the top band with positive oriented crossing and the bottom band with negative oriented crossing. This can be seen in Figure 2.4. Invoke your imagination and creative visualization. To see the tube, fix the lower surface on the left of Figure 2.4 and pull the upper surface to the right. The striped side of the surface forms the outside of the tube and the gray side forms the inside of the tube as in Figure 2.6. Then $\uparrow \pm \uparrow = |T|$. $|T| = \uparrow \mp \uparrow$ is proved similarly by viewing Figure 2.4 from the other side of the plane. Notice the K and K' on the far edges of the tube. This is to remember that there are or can be additional bands attached to the surface, which will be important. Without K and K', "|" are disks, and the local surface of the picture forms a simple annulus. □

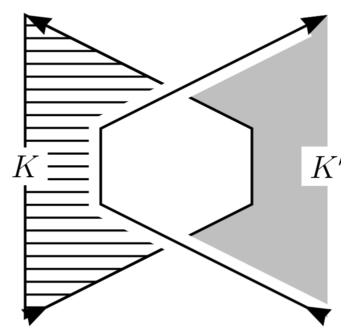


Figure 2.5: Two Seifert circles connected by bands with opposite twists

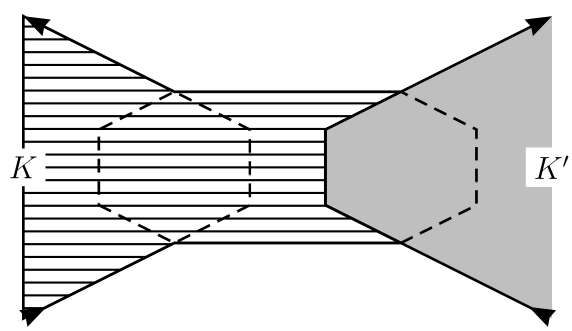


Figure 2.6: Stretching the twisted bands to see the tube.

2.0.4 Four relations between algorithmic surfaces.

Lemma 2. *The following four relations hold:*

1. $\Big\uparrow + \Big\uparrow \substack{?\\+} \Big\uparrow \sim \Big\uparrow \substack{+\\?} \Big\uparrow + \Big\uparrow$ 2. $\Big\uparrow \substack{?\\+} \Big\uparrow + \Big\uparrow \sim \Big\uparrow + \Big\uparrow \substack{+\\?} \Big\uparrow$

3. $\Big\uparrow \substack{?\\-} \Big\uparrow - \Big\uparrow \sim \Big\uparrow - \Big\uparrow \substack{-\\?} \Big\uparrow$ 4. $\Big\uparrow - \Big\uparrow \substack{?\\-} \Big\uparrow \sim \Big\uparrow \substack{-\\?} \Big\uparrow - \Big\uparrow$

In the original paper by Bar-Natan *et.al.* [8], the authors make the argument of equivalence by addressing Reidemeister moves on braids. However, this adds some confusion, for the proof regards surfaces and are best treated as such. For lack of a better place to begin, how about proving Relation 1.

Proof. Consider Figure 2.7. This is a display of Relation 1. The diagram shows the Seifert circles with the appropriately attached bands and orientation. Imagine the two dark bands highlighting a section of the boundary of the surface as tracks. One track begins at the bottom left corner of crossing A and runs the over crossing portion of crossing B, then along the boundary of K to some point past crossing C. The second track begins at the bottom right of crossing A, follows the boundary of K' toward crossing C, travels the over-crossing of C, and continues along the boundary of K'' to its final position. Slide and pull crossing A. The surface is pliable and will bend and roll as A moves along the tracks while preserving orientation. Crossing "A" is now isotoped along the tracks from its beginning position A to its ending position D. Case 1 is an isotopy. Case 3 is shown in Figure 2.8. Again, the dark lines along the boundary represent the tracks for the isotopy. Relation 3 is proved similarly to Relation 1 as seen in Figure 2.8. Notice that the picture proves Relation 3 from right to left. Thus, we have the desired results for Relation 1 and 3.

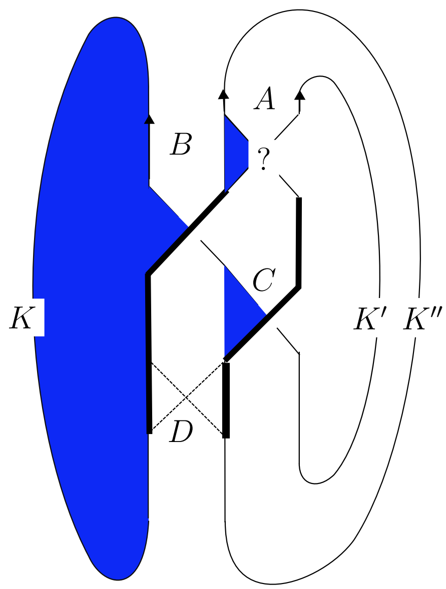


Figure 2.7: Relation 1

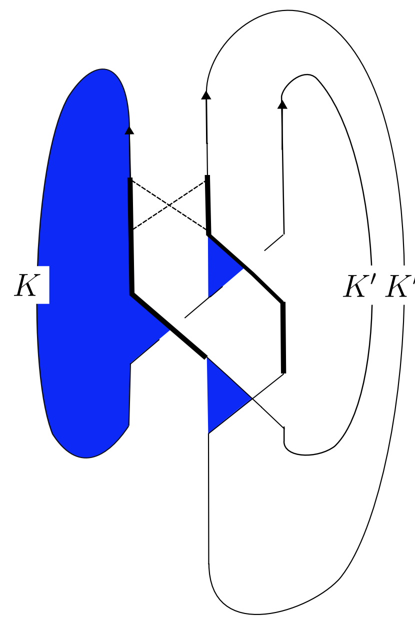
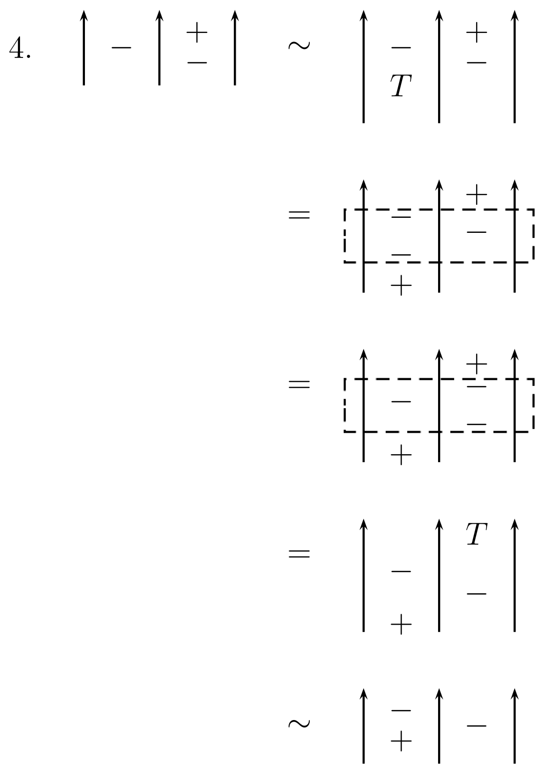


Figure 2.8: Relation 2

When Relation 2 and Relation 4 are all of the same signs, then the proofs proceed as Relation 1 and Relation 3 and they are isotopic. Otherwise, Relation 2 and Relation 4 require tube-equivalence, conjecture the authors of [8]; however, I believe they are all cases of isotopy based on the authors' own argument that when Relation 2 is all "+", this is an isotopy. The only difference is that the tracks are on the underside of the crossing instead of over the crossings. Thus, "?" proceeds the same as Relations 1 and 3, but on the underside and tube-equivalence is not necessary.

Relation 4 can be proved using tube-equivalence applications of Lemma 1 and Lemma 2, Relation 3. Relation 2 can proved similarly to Relation 4. Then,



Tube-equivalence,

By Relation 3,

Lemma 1,

Tube-equivalence, as desired.

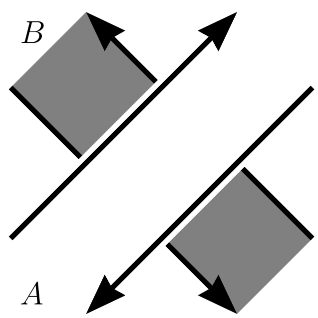
2.0.5 Band-handle form and algorithmic surfaces.

There are two steps left to complete this proof. Lemma 3 will prove that a surface in band-handle form is tube-equivalent to the algorithmic surface of the corresponding projection. Finally in Lemma 4, we will see that if L_1 and L_2 are link projections of L, then the algorithm surfaces of L_1 and L_2 are tube equivalent.

Lemma 3. *A surface in band-handle form is tube-equivalent to an algorithmic surface.*

Proof. Let S_1 and S_2 be spanning surfaces of L. Show that any two algorithmic surfaces are tube-equivalent. S_1 and S_2 can be presented in band-handle form. If a band crosses over another band as in Figure 2.9, "Crossing of oriented bands", then application of the Seifert algorithm inserts a tube between the bands. Because of the opposing orientations, the Seifert circles always lie between the edges of the bands and never in the complement. This works since the gray side of band A matches with the gray side of band B. Thus the outside of the tube is gray and the inside is white after application of the Seifert algorithm. Inspect Figure 2.10. Imagine lifting the white, upper bands, and the gray bands staying fixed. The tube forms in this way and will "round-out" as the surface is lifted off the plane.

If bands are such that the orientations do not match, then make a half-twist to one band as seen in Figure 2.11. Then the previous applies. Thus, a surface in band-handle form is tube equivalent to an algorithmic surface. □



Crossing of oriented bands.

Figure 2.9: Crossing of oriented bands.

2.0.6 Tube-equivalence of algorithmic surfaces.

Lemma 4. *Let L and L' be different projections of the link L, then the algorithmic surfaces are tube-equivalent.*

Proof. The second part of this proof is to consider how Reidemeister moves affect

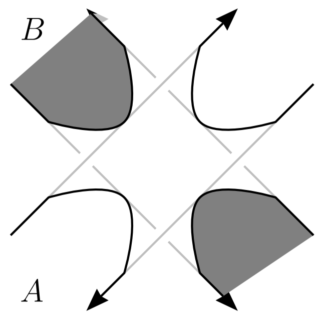


Figure 2.10: Seifert algorithm applied to band-handle projection.

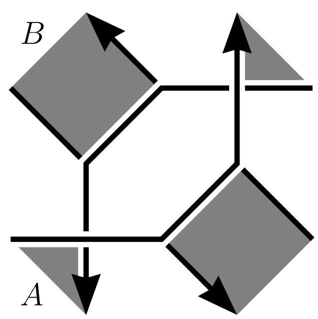


Figure 2.11: Add a half twist to orient surface.

the algorithmic surface. Each Reidemeister move affects the projection locally, inside a ball. It is sufficient to consider how each R move affects the projection surface. In each case fix the projection of L, and apply the Seifert algorithm. Next, apply the Reidemeister move in consideration to L and call the new projection L'. Apply the Seifert algorithm to L'. Each surface will be either isotopic or tube-equivalent. In the diagrams that follow, K, K' and K'' are the portions of L or L' that are unaffected by the local move, but we must not forget that there is *more surface* so as to not oversimplify.

$R1$ is a simple isotopy. If one could take the edge of the existing surface, pinch it, give it a half twist. The $R1$ move will pull back to being to the original disk. There is no change to the surface. Figure 2.12

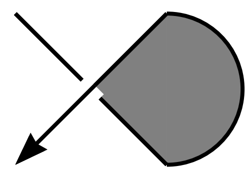


Figure 2.12: $R1$ is an isotopy on surfaces.

$R2$ has three cases to consider. These depend on the orientation of L and where the $R2$ move happens in relation to the link.

Case 1. See Figure 2.13. In the left diagram K and K' are connected by a band. After the $R1$ move and application of the Seifert algorithm, the band has a half twist. Thus, $R1$ **Case 1** is an isotopy.

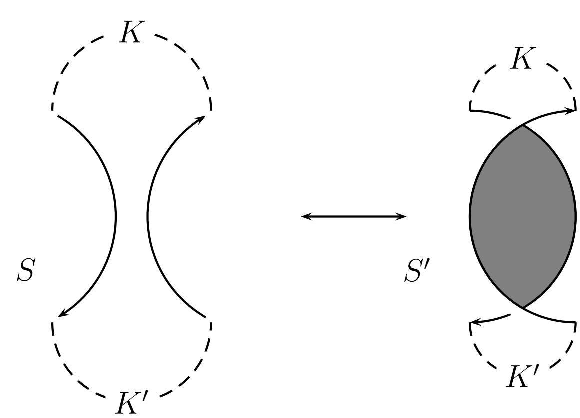


Figure 2.13: Seifert algorithm creates a twisted band between K and K'

Case2. Examine Figure 2.14. In the first diagram, we have fixed a link L with the noted orientation and show $R2$ applied to L. Notice that K and K' represent those portions of L not affected by $R2$ but which will become part of the surface when the Seifert algorithm is applied to the L. In the second diagram the Seifert algorithm has been applied to L. This is an immediate application of **Lemma 1**, i.e., one positive crossing and one negative crossing. Thus, Case 2 is tube-equivalence.

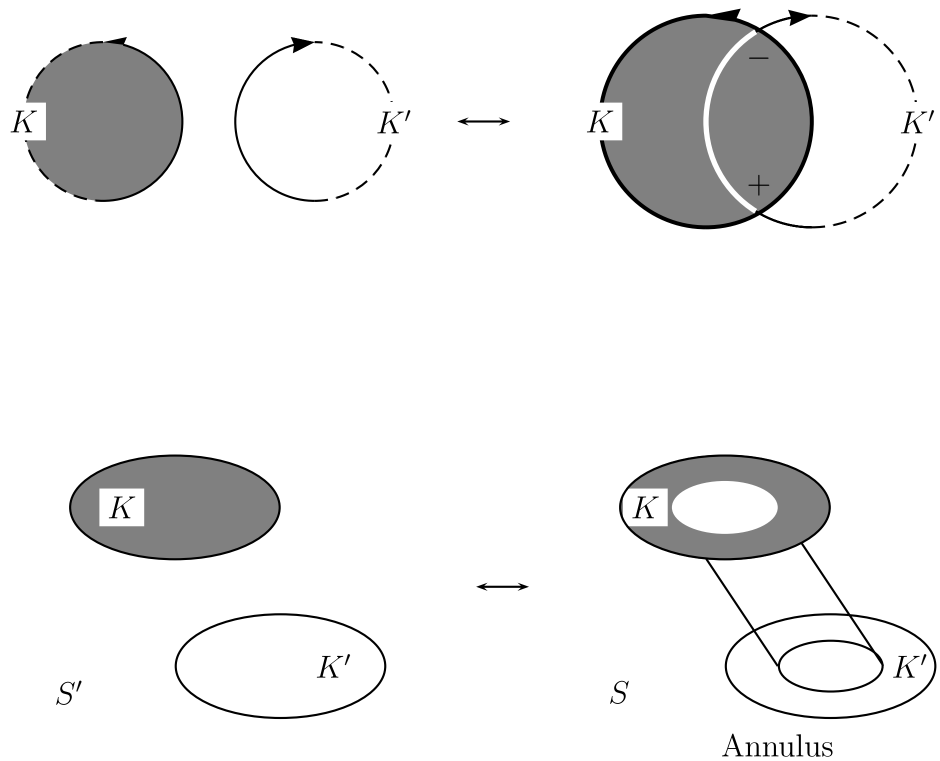


Figure 2.14: $R2$ changes the surface by a tube. The bottom line is the 3D view on the surface.

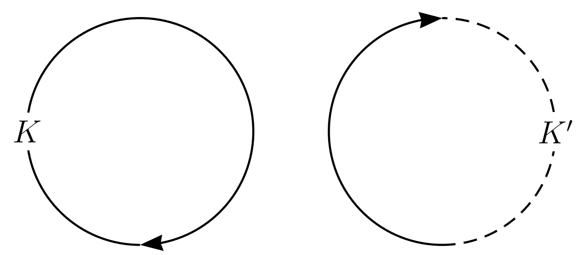
Case 3. See Figure 2.15. After applying Seifert algorithm the middle Seifert circle can be flipped, Close inspection reveals that a tube has been added between SK and SK'. Thus, by Lemma 1 these are tube equivalent.

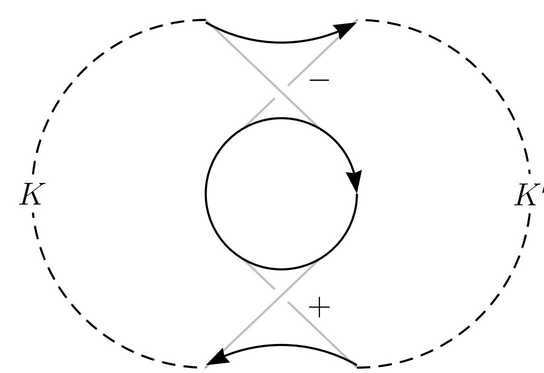
R3. Because of the different orientations there are four cases to investigate. In the diagrams that follow the dotted circle surrounding each crossing represents a small sphere in which all ambient isotopies and addition and removal of tubes will occur. See Figure 2.16.

Cases 1 through 3 proceed in a similar fashion. Case 4 requires additional steps.

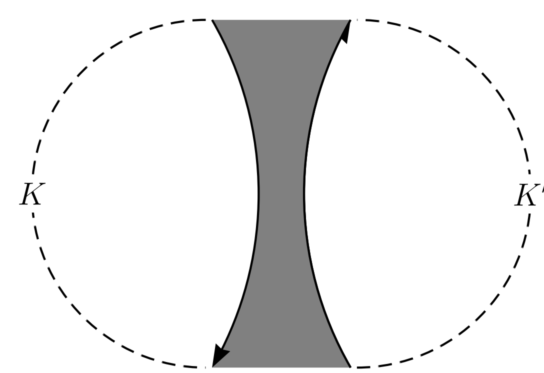
Apply the Seifert algorithm to Case 1, Case 2, and Case 3, and use the notation of "|" for the Seifert circles and "+" and "−" for the crossings. The results of the Seifert algorithm are seen in Figure 2.17. This gives us a schematic that matches that used in Lemma 1 and Lemma 2. Comparing the algorithmic surface we can move from one to the other by applications of Lemma 2. Thus $R3$ is either an isotopy or tube-equivalence depending on which case of Lemma 2 is used.

Now prove $R3$ Case 4. Apply the Seifert algorithm to Case 4. Figure 2.18 illustrates why an immediate application of Lemma 2 does not work. Figure 2.19 shows the process by which we can track the completion of $R3$ with this orientation. Proceed as follows. There are two $R2$ moves made. The first $R2$ is either $R2$, Case 1 (Figure 2.13) or $R2$, Case 3 (Figure 2.15). The second $R2$ is $R2$, Case 2 (Figure 2.14). The orientations are what determine the appropriate case. The next $R3$ move is $R3$, Case 1. The next two R moves proceed in the reverse order of the first two. Thus, the algorithmic surface is tube-equivalent under $R3$, Case 4; and we have proved that S_1 and S_2 are tube equivalent. □



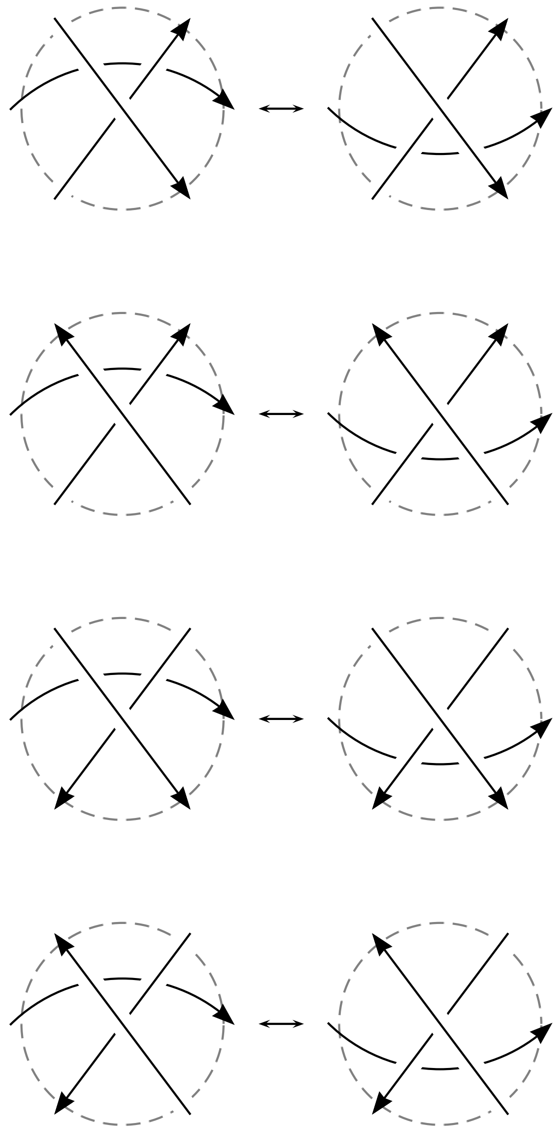


Algorithmic surface applied after $R2$.



Twist the band to see the tube.

Figure 2.15: $R2$ Case 3 shows tube-equivalence.



Case 1.

Case 2.

Case 3.

Case 4.

Figure 2.16: The four possibilities for $R3$.

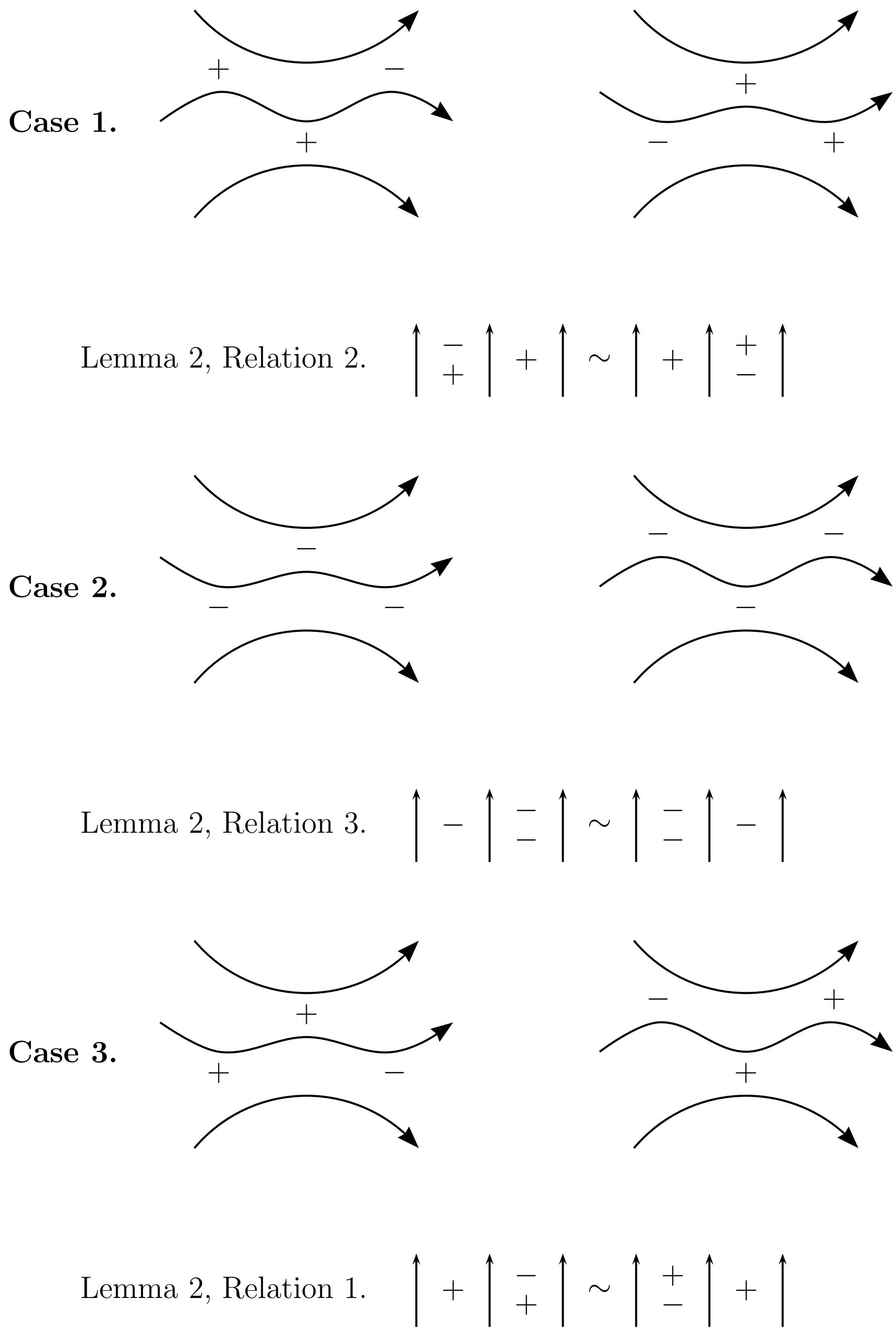


Figure 2.17: $R3$ on the algorithmic surface.

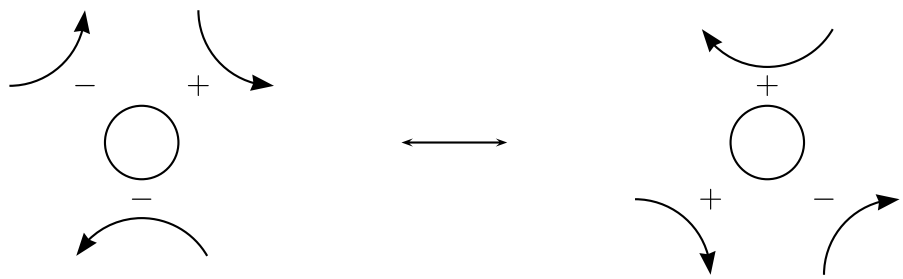


Figure 2.18: Seifert algorithm applied to $R3$, Case 4.

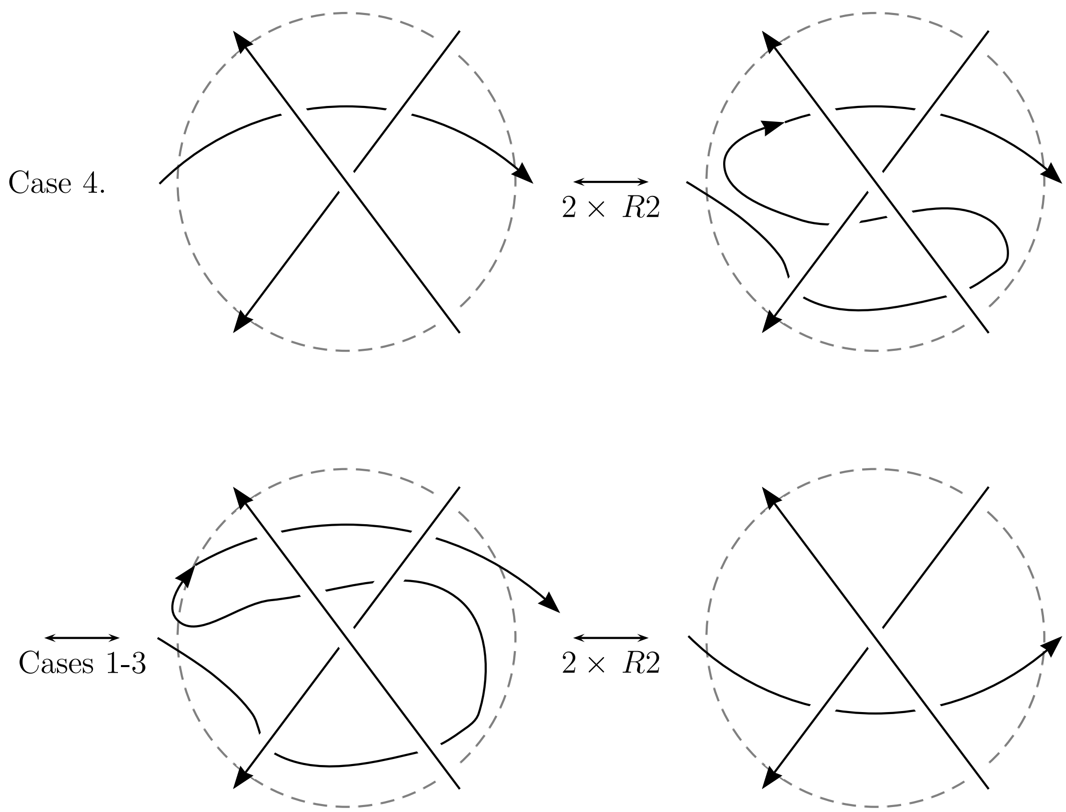


Figure 2.19: Tube-equivalence for $R3$ Case 4.

CHAPTER 3

Tube-equivalence of Seifert surfaces.

In proving Theorem 1 no connectivity assumptions were made. Because of this, the fact that the Alexander Polynomial is a link invariant does not follow immediately from tube-equivalence (Sec. 3.2).

3.1 Tubes and connectedness.

Recall that a Seifert surface is a connected spanning surface. If we require the spanning surfaces to be connected, then tube-equivalence will mean that *all* the surfaces in a tubing/untubing sequence connecting the surfaces remain connected.

Definition 3. Seifert surface S_1 and S_2 are *tube-equivalent* if they are related by isotopy and sequences of adding and deleting fillable tubes that never disconnect the surface.

Theorem 2. *Two Seifert surfaces S_1 and S_2 with the same boundary L are tube-equivalent.*

When discussing connectedness and tube interference below, note that for each projection P there is an algorithmic surface S. If we change P by a Reidemeister move R' this will result in the new projection P' with corresponding algorithmic surface

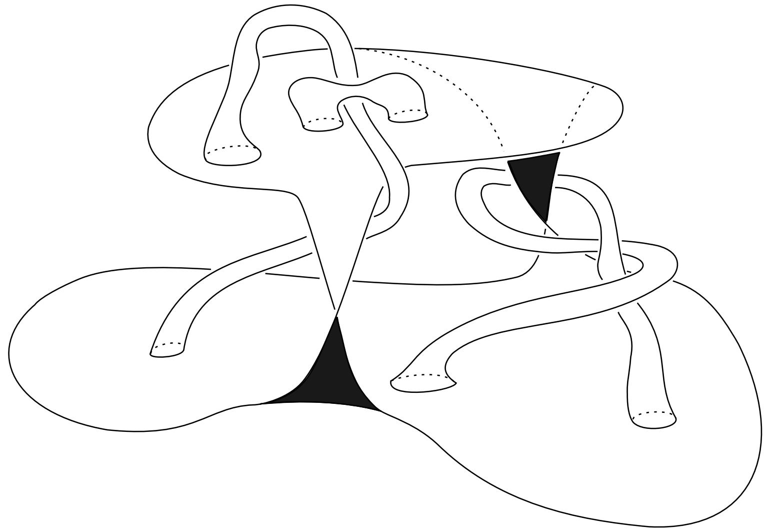


Figure 3.1: A Devious Tuber

S'. Theorem 1 proceeded at the level of the projections. The proof of Theorem 2 we will give proceeds at the level of surfaces. On the projection level $R2$ could connect or disconnect the projection and hence the associated algorithmic surface.

We would like to state, without proof, the following theorem.

Theorem. *If P_1 and P_2 are two connected link projections that are related by a sequence of Reidemeister moves (equivalently, the links corresponding to P_1 and P_2 are isotopic), then there exists a sequence of Reidemeister moves through connected diagrams.*

Remark: The Theorem can be proved by fixing an arc on a projection. Throughout the sequence of Reidemeister moves let there be a ball around the arc. As R moves are made to the link, any that disconnect require an $R2$ from within the ball to the disconnected link component to connect. Even though this is a stronger result than that for surfaces we will give the tube-equivalence directly for surfaces. In our work we will proceed differently.

Obviously, handles can be added and deleted locally because of necessary global isotopies, but we do not see how this process affects the surface for upcoming R moves. Adding tubes might be necessary to establish the relation between surfaces following Definition 3. We consider each R move individually. *Note that adding tubes to solve a connectedness problem can create a problem for the following moves.* The problem that a tube restricts isotopy of the surface in accordance with upcoming R moves is called tube-interference. Thus, at each crossing there are two points to consider to maintain Seifert surfaces: First of course, connectedness. In order to maintain connectedness tubes will have to be attached. This requires carefully checking that it does not interfere with the algorithmic construction. We call this tube-interference.

3.1.1 Connectedness.

If a Reidemeister move on a diagram P disconnects, then the algorithmic suface of P' will be a disconnected surface S', Figure 3.2(C); hence, not a Seifert surface. We can correct this by first adding a tube to the algorithmic surface of P. Call the resulting surface $\tilde{S}$. See Figure 3.2(B). Now proceed with R' and apply the Seifert algorithm to P'. The algorithmic surface of P' will differ from the algorithmic surface of P by a tube, Figure 3.2(C). By Theorem 1, the algorithmic surfaces are tube equivalent by addition of a handle as in $\tilde{S}'$, Figure 3.2(D).

3.1.2 Tube Interference.

As one can see in the picture of "A Devious Tuber", Figure 1, surfaces can become quite complicated, and the addition and deletion of tubes from R to R may only serve to aggravate the situation. Tube interference is when a tube on the surface prevents the corresponding tube-equivalence that the next R dictates. Considering

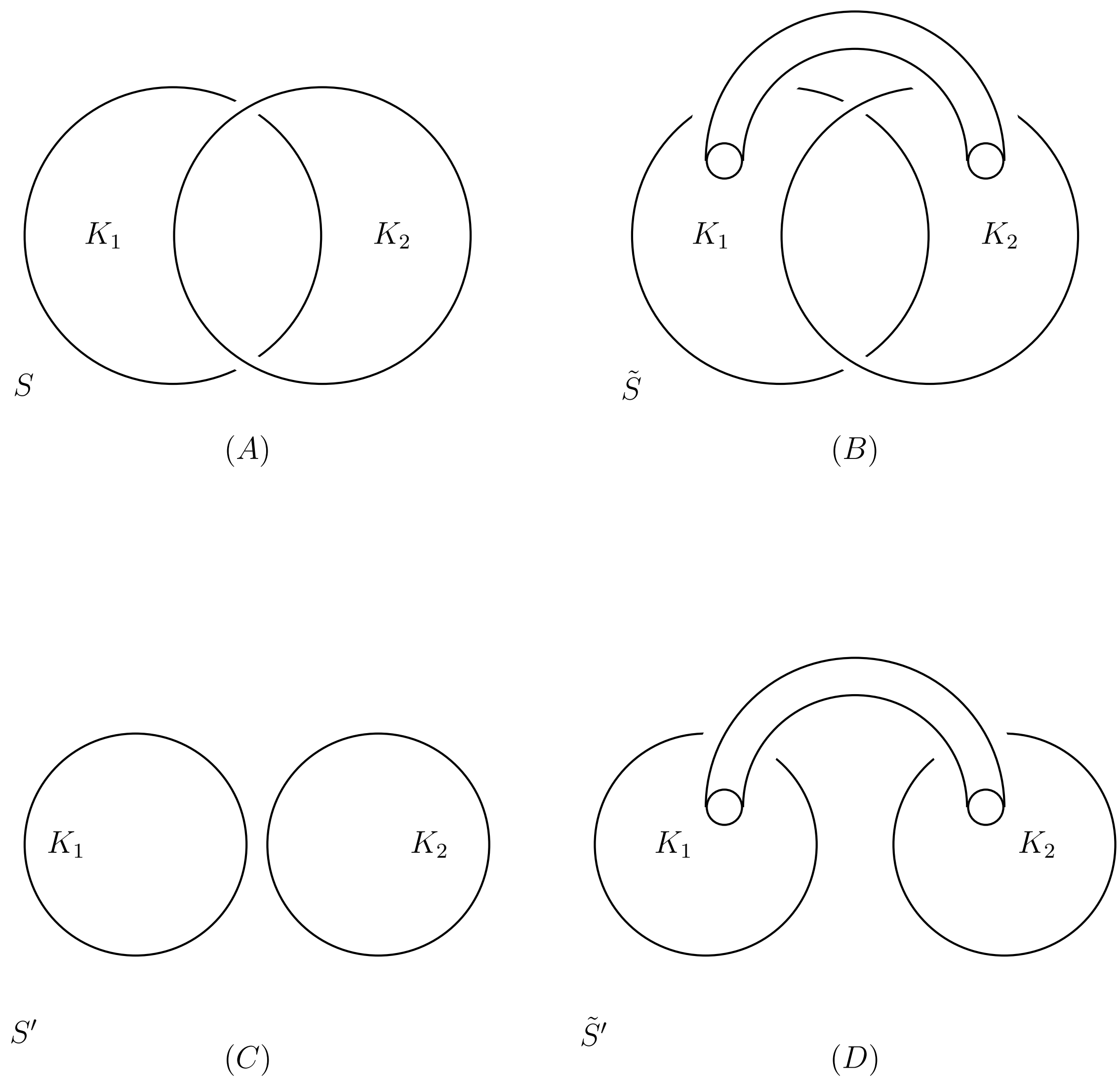


Figure 3.2: Maintaining connectedness.

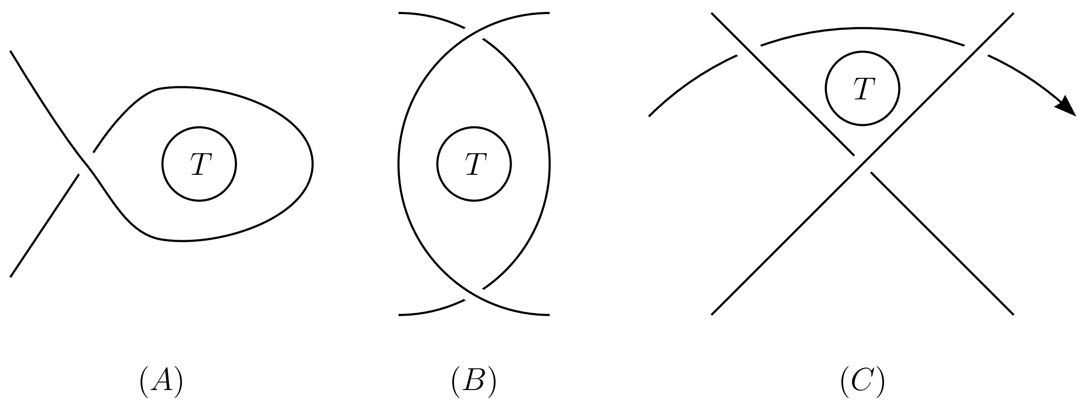


Figure 3.3: Tube-interference.

only the projection there is no problem, but at the surface level there could be. When a tube runs through that part of 3-space where the R move takes place, tube-interference could prohibit tube-equivalence. As illustrated in Figure 3.3 three cases occur according to which Reidemeister moves is involved.

3.1.3 R1 and tube interference.

Figure 3(A) can have two cases as seen in Figure 3.4. In the left diagram the surface is on the inside of the corresponding projection, and the tube T terminates on the surface where it is seen in the diagram. Slide T along the twist and it will end on the left side with orientation preserved. Then the area where the tube was attached can be contracted into the rest of the surface. See Figure 3.5.

The second case, Figure 3.4 on the right, is where the surface is along the outside of the boundary of L and T passes through the empty loop. The loop cannot be contracted without compressing T; hence prohibiting $R1$. To remedy this, first add a tube T' to S that attaches up to isotopy where T attaches, but make sure that T' passes through some other part of space avoiding a neighborhood of a disk bounding

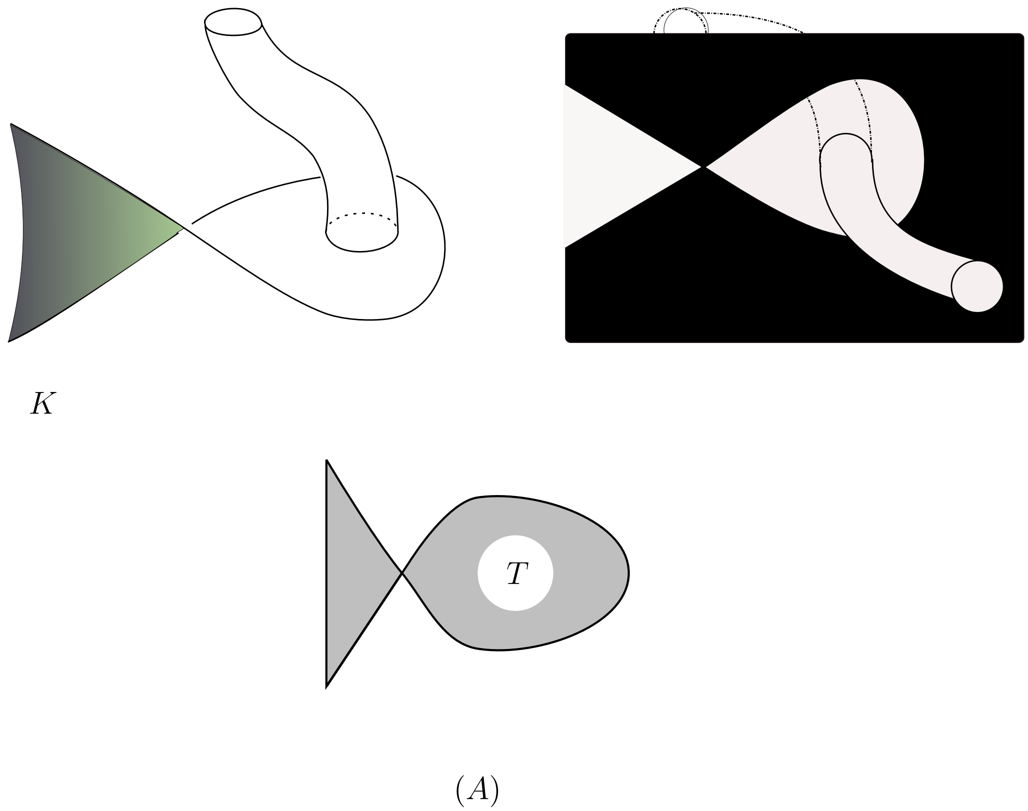


Figure 3.4: $R1$. The shaded regions represent the surface.

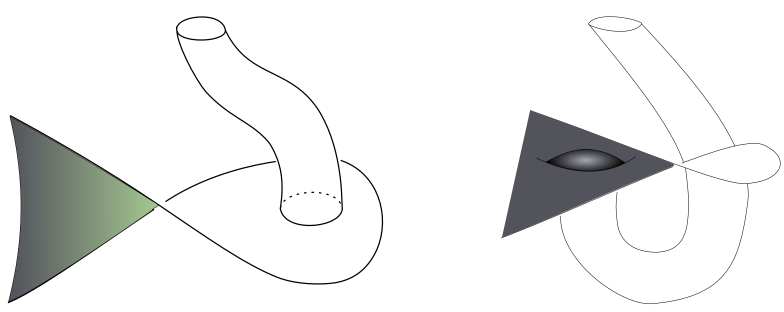


Figure 3.5: $R1$. The pictures are isotopic.

the loop and maintaining proper orientation. Compress T. The loop may now be contracted corresponding to $R1$. Then S and S' differ by T'. By Theorem 1 add a tube to the algorithmic surface to preserve tube-equivalence.

3.1.4 R2 and tube interference.

Consider $R2$ as in Case 1 from **Theorem 1** as show below in Figure 3.6.

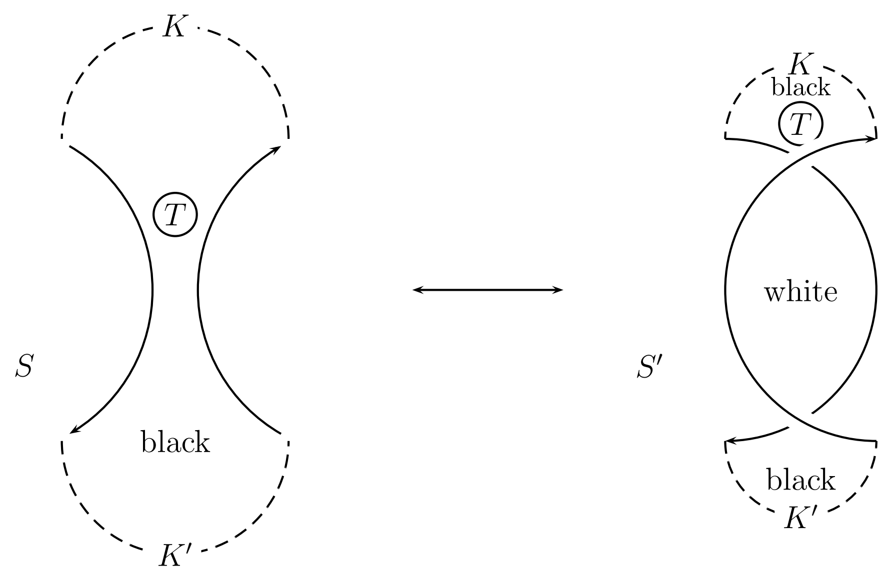


Figure 3.6: Siefert algorithm creates a twisted band between K and K'

S^j has a band between parts of surface S, K and K'. $R2$ has the effect of making a half twist along the band as in S', Figure 3.6. T can be isotoped out of the way of the band as seen in Figure 3.6(B), or it can be pulled around with the twist. In this situation $S = S'$.

Next consider $R2$, Case 2. Case 2 changes the algorithmic surface by attaching nested Seifert circles by a tube as shown in Figure 3.7.

What happens to the surface when there is a tube within the tube? At the level of P, $R2$ is a simple move. But S has the tube passing through the annulus in Figure 3.7

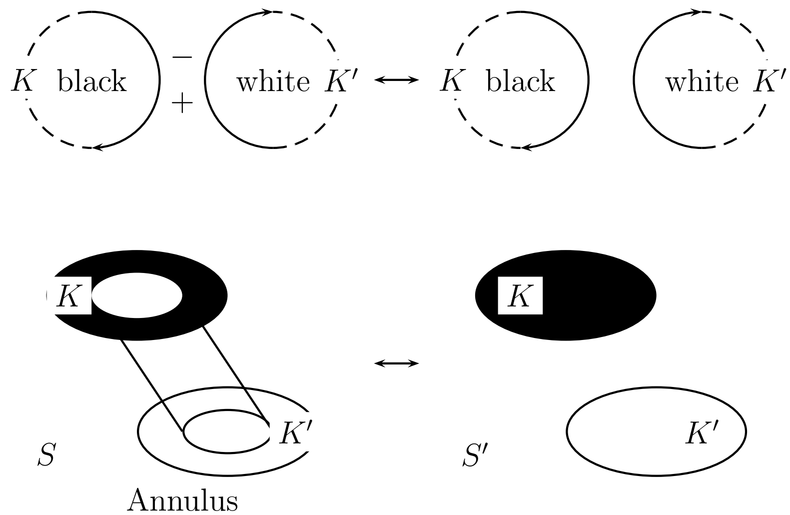


Figure 3.7: $R2$ changes the surface by a tube. The top line is a projection. The bottom line is the 3D view on the surface.

(Note: the tube inside the annulus is not in the picture). There could be a disconnect of K and K' by the $R2$, and the inside tube must be removed. There are two steps to move from S to S'. First add a tube that preserves orientation, does not pass through the annulus, and attaches up to isotopy where the tube passing through the annulus does. Next, if $R2$ disconnects, attach a tube from K to K' so that S' stays connected. Apply $R2$. Apply the Seifert algorithm to S and S'. By Theorem one shows $S \sim S'$.

$R2$ Case 3 is an example of a tube attaching surfaces as in 3.2(D). Thus a tube T passing through the connecting tube prevents $R2$. Where T attaches to the surface add a new tube T' that attaches up to isotpy where T attaches, but goes around or over the crossing. Delete T. Further, there is a chance that $R2$ moves disconnect the link. Before $R2$ attach a tube so that S' remains connected. Comparing the algorithmic surfaces and Theorem 1 indicates they are tube-equivalent.

3.1.5 R3 and tube interference.

There are four cases of $R3$ based on the orientation of L that are solved in Theorem 1. Now each case deals with a tube in the middle of the crossing as seen in 3.3. With T passing through the middle of the crossing as it is related to the surface, proceed as above by first adding tube(s) to preserve connectedness, then compressing the tube interfering with the crossing.

3.1.6 Proof of Theorem 2.

Proof by induction. The proof will proceed by constructing a sequence of surfaces $S^0, \ldots, S^k$ associated to each $P^0, \ldots, P^k$. Each surface will be constructed by applying the Seifert algorithm to the projection $P^0, \ldots, P^k$ and adding tubes. We know there exists some sequence of R moves $\{R_j\}_{j=1}^k$ from P_1 to P_2. Let P^j indicate the projection of L after the R_j move such that $P_1 = P^0$ and $P_2 = P^k$. Similarly, S^j is the associated algorithmic Seifert surface of P^j. This gives the following diagram.

$$\begin{array}{ccccccccccc} P_1 = & P^0 & \xrightarrow{R_1} & P^1 & \xrightarrow{R_2} & P^2 & \xrightarrow{R_3} \cdots \xrightarrow{R_{k-1}} & P^{k-1} & \xrightarrow{R_k} & P^k & = P_2 \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\ & S^0 & & S^1 & & S^2 & & S^n & & S_2 & \end{array}$$

Let S_1 and S_2 be Seifert surfaces bounding the link L. S_1 can be isotoped to band-handle from. Apply the Seifert algorithm. $P_1 = P^0$ is connected and S^0 is a Seifert surface. Apply the Seifert algorithm to P^1. Proceed as follows: Compare algorithmic surfaces. If a P^1 is connected, take the usual algorithmic surface. If P^1 is not connected, take the usual algorithmic surface, and add an arbitrary tube(s) to make it connected. Call the resulting surface S^1. Depending on R we can correct a disconnect or tube-interference as described above. Thus $S^0 \sim S^1$.

Suppose we have constructed the Seifert surface S^{k-1} such that S^{k-1} is the algorithmic surface of P^{k-1} with tubes attached that account for the various tube interferences and disconnects along the sequence of R_j. We construct S^k such that S^k is the algorithmic surface of P^k with handles attached and $S^{k-1} \sim S^k$. Assume that R_k changes P^{k-1}, respectively. Then there is a corresponding projection P^k that differs from P^{k-1} by R_k. Compare the algorithmic surfaces of P^{k-1} and P^k. The R_k is a local change to P^{k-1}. We can assume R_k happens inside a 3-ball, and following theorem 1 there will be tube-equivalences inside the 3-ball. If R_k disconnects P^k, then the algorithmic surface of P^{k-1} and P^k differ locally by a handle. Attach a handle to connect. The resulting surface is $S^{k-1} \sim S^k$. If R cannot translate directly from S^{k-1} to S^k because of tube-interference, then by **Tube-interference** add and delete the necessary handles. The result is $S^{k-1} \sim S^k$. Thus, when we have followed all S^j's through the P^j's and arrive at S_2. S^k resembles the algorithmic surface for S_2, but it is guaranteed to be connected and, most likely, has additional tubes attached, thus S^k is a Seifert surface, and $S^{k-1} \sim S^k$. Then by induction we have

$$S_1 = S^0 \sim S^1 \sim \cdots \sim S^{k-1} \sim S^k = S_2 \text{ , as desired.}$$

3.2 Conclusion.

One reason for the need to consider connectedness is the relation to the Alexander polynomial Δ_L. The Alexander polynomial can be defined from a Seifert surface using the Seifert pairing [1]. The Seifert pairing is defined by a matrix of linking numbers between oriented cycles on the surface and pushout of those cycles into the compliment of the surface. Let S be a Seifert surface. Let $f : S \times [-1, 1] \to \mathbb{R}^3$ be a homeomorphism such that $f(S \times \{0\}) = S$, $f(S \times \{1\})$ lift cycles off of S in the positive direction, and let $f(S \times \{-1\})$ lower cycles off of S. The set of cycles is the first homology of S. Then for $a \in H_1(S)$, $f(S \times 1) = a^+$ and $f(S \times 0) = a$. This takes cycles that may intersect on S and creates cycles and links of the cycles. These may be linked, and that is what we use to find the linking numbers. Thus, define the following mapping:

$$\begin{aligned} \theta : H_1(S) \times H_1(S) \to& \mathbb{Z} \\ \theta(a,b) =& \mathrm{lk}(a, b^+) \\ \text{such that } a, b \in H_1(S) \end{aligned}$$

This gives the Seifert pairing of the embedded surface.

How does tube-equivalence affect the Alexander polynomial? What if we have a disconnected spanning surface then connected it with a tube? Tube-equivalence according to Definition 2 can *fool* Δ_L. In fact, if two *Seifert* surfaces are tube-equivalent, then their corresponding Seifert matrices are called S-equivalent. A well written proof of S-equivalence can be found in [6]. But a heuristic explanation is provided here.

A priori one can define a polynomial using the Seifert pairing from any spanning surface. But it turns out that this is not the Alexander polynomial if the surface is not connected. This can be seen by considering, e.g., a split union of two figure-eight knots 8_1 and 8_2. First calculate the Seifert pairing of a connected surface to obtain the Alexander polynomial, as correctly calculated using the Seifert surface. This can be seen by constructing the Seifert matrix.

$$M(8_1 \cup 8_2) = \left(\begin{array}{c|c|c} M(8_1) & 0 & 0 \\ \hline 0 & M(8_2) & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \Rightarrow \det(M(8_1 \cup 8_2)) = 0$$

The last row and column of zeros come from the generator given by the meridian of the tube connecting 8_1 and 8_2. It always has a linking number of zero with all other cycles.

In contrast, if we were to simply calculate the matrix obtained by the Seifert pairing on the spanning surface of 8_1 and 8_2, we obtain the following matrix.

$$M(8_1 \cup 8_2) = \left(\begin{array}{c|c} M(8_1) & 0 \\ \hline 0 & M(8_2) \end{array}\right) \Rightarrow \det(M(8_1 \cup 8_2)) = \Delta_{8_1 \# 8_2}(t) \neq 0$$

The polynomial calculated from a split union of the Seifert surfaces of the knots yields the Alexander polynomial of the connected sum of the two figure-eights, which is not zero.

The reason for the discrepancy between the two results is that when a tube is added to a connected surface L, the tube creates two new generators of the first homology in the Seifert matrix. The generator given by the meridian of the tube has a linking number of zero with all other cycles. The linking number of the generator along the longitude may, very likely, have non-zero linking numbers. Making a change of basis to the homology, we obtain the following matrix.

$$M(L) = \left(\begin{array}{ccc|cc} & & & * & 0 \\ & M(L) & & \vdots & \vdots \\ & & & * & 0 \\ \hline 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right)$$

Then $\Delta_L(t)$ can be calculated from the equation $\Delta_L(t) = \det\left(A - tA^T\right)$.

Thus the restriction of connectedness on spanning surfaces has allowed us to correctly calculate the $\Delta_L(t)$ for any Seifert surface, and we see that tube-equivalence does not change the Δ_L, but enables one to study surfaces of links without changing the underlying structure.

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