Hyperbolic Surface-Arc Complements

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Abstract

We show that the fundamental group of a prime alternating surface-arc complement is δ-hyperbolic in case the genus of the surface is greater than zero.

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1 Introduction

Let F be a closed orientable surface of genus g with one boundary component. Let $\hat{F} = F \times [-1, 1]$ be the fattened up surface and let A be an embedded arc in $\hat{F}$ with $\partial A \subseteq \partial F \times (-1, 1)$. The manifold $\bar{M}$ is obtained from $F \times [-1, 1]$ by removing an open neighborhood of the arc A. Note that $\bar{M}$ is a closed orientable manifold with boundary consisting of a surface of genus $2g + 1$. We call $\bar{M}$ a *surface-arc complement of genus g*. Note that in case $g = 0$ the surface F is a disc and the surface-arc complement is a 3-ball with a properly embedded arc removed. This is the setting of classical knot theory. We say that a surface-arc complement $\bar{M}$ of genus $g > 0$ is *prime* if, whenever we have an embedded 3-ball B in $\bar{M}$ such that $S = \partial B$ intersects A in exactly two points, then $B \cap A$ is an unknotted arc in B, that is $\pi_1(B - B \cap A) = \mathbb{Z}$.

Let $\pi : \hat{F} \to F$ be the obvious projection. Throughout this paper we assume that $\pi(A)$ does not contain trivial crossings and that $F - \pi(A)$ is a collection of open discs, i.e. $\pi(A)$ is the 1-skeleton of a cell decomposition of F. In case F is a surface of genus $g > 0$ we call $\pi(A)$ *prime* if each disc $D \subseteq F$ with boundary ∂D meeting $\pi(A)$ in just two non-double points does not contain any crossings. Note that this implies in partiular that the arc runs across some handle. In case $g = 0$ we call A *prime* if each disc $D \subseteq F$ with boundary ∂D meeting $\pi(A)$ in just two non-double points either contains no crossing or all of them.

Menasco [4] has shown that in the classical situation where F is a disc and A is an alternating arc in the 3-ball $F \times [-1,1]$ such that $\pi(A)$ is prime, then the interior of the arc complement $\bar{M}$ admits a co-finite volume hyperbolic structure unless the fundamental group is a torus knot group. We prove an analogous result in the higher genus setting.

Theorem 1.1 *Let A be an alternating arc in $F \times [-1,1]$, where F is a surface of genus at least one. Assume $\pi(A) \subseteq F$ is prime. Then the surface-arc complement $\bar{M}$ is prime and its interior M admits a convex co-compact hyperbolic structure. In particular the fundamental group $\pi_1(\bar{M})$ is δ-hyperbolic.*

Projections of surface-arcs and knots are sometimes called virtual arcs and knots (see Kauffman [3]). Our interest in surface-arc complements comes from connections with ribbon-disc and 2-knot groups. It is shown in [2] that the homology 3-manifolds obtained from a surface-arc complement by coning off the top surface $F \times \{1\}$ is homotopically equivalent a ribbon-disc complement in the 4-ball. In [1] it is further shown that under certain conditions hyperbolicity of the surface-arc complement survives the coning process. This leads to many examples of δ-hyperbolic ribbon-disc groups.

2 Topology

Let M be the interior of the manifold $\bar{M}$ of Theorem 1.1. We think of M as a line complement in a fattened surface $\hat{E} = E \times [-1,1]$, where E is a plane with at least one handle. Let L be an embedded line in $\hat{E}$, which is alternating with respect to the projection $\pi : \hat{E} \to E$. We assume that the projection $\pi(L)$ has no trivial crossings, that the complement $E - \pi(L)$ is a collection of simply connected regions, and that it is prime (a disc in E with boundary meeting $\pi(L)$ in just two non-double points does not contain any crossings).

Let $E_0 = E \times \{0\} \subseteq \hat{E}$. We position L so that all crossings are contained in a fattened disc $D \times [-1,1]$, $D \subseteq E$. We further assume L lies on E_0 except near crossings of L, where it lies on a bubble. See Figure 1. Let E_+ (E_-) be E_0 with each disc of E_0 inside a bubble replaced by the upper (lower) hemisphere of that bubble. Let $\hat{E}_+$ be the upper part of $\hat{E}$ bounded from below by E_+. Similarly we define $\hat{E}_-$.

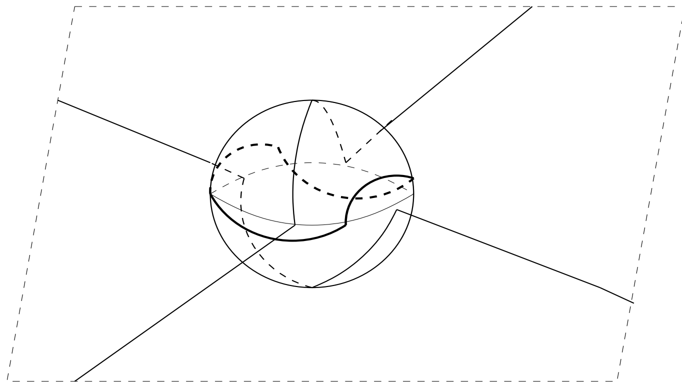

Figure 1: A bubble with a saddle.

Let S be a closed surface embedded in $\hat{E}$. We may assume each intersection with a bubble is a saddle shaped disc σ (see Figure 1). Note that $\partial\sigma$ consists of four arcs, two of which lie in E_+ and two lie in E_-. We refer to the former as positive $\partial\sigma$-arcs and the latter as negative $\partial\sigma$-arcs. Consider $\mathcal{C}_+ = E_+ \cap S$ and $\mathcal{C}_- = E_- \cap S$. We assume that S is in general position so that both intersections are finite sets of disjoint circles on S. We also assume that the total number of circles in $\mathcal{C}_+ \cup \mathcal{C}_-$ can not be reduced within the isotopy class of S.

Lemma 2.1 *Suppose $C \in \mathcal{C}_+$ and that C bounds a disc D' in S. Assume furthermore that C is innermost, that is it bounds a disc D in E_+ that does not contain a curve from $\mathcal{C}_+$ in its interior. If C crosses two successive bubbles as in Figure 2, then S contains a circle isotopic in M to a meridian of L.*

Proof: We argue as in Menasco [4]. Alternating implies that the intersection of L with the bubbles must lie on opposite sides of C (see Figure 2). Hence one of the two intersection arcs of L with a bubble H must lie in D. See Figure 2. It follows that $C = \partial D$ intersects a bubble H at least twice and that there is a saddle σ so that the two positive $\partial\sigma$-arcs of $H \cap \sigma$ both lie on $C = \partial D$. Let b be the band $H \cap D \subseteq E_+$. Then the annulus $b \cup \sigma$ contains a circle isotopic in M to a meridian of L. Hence the surface S also contains such an an annulus $b' \cup \sigma$, where b' is a band in D' (the union of discs $D \cup D'$ bounds ball in M). •

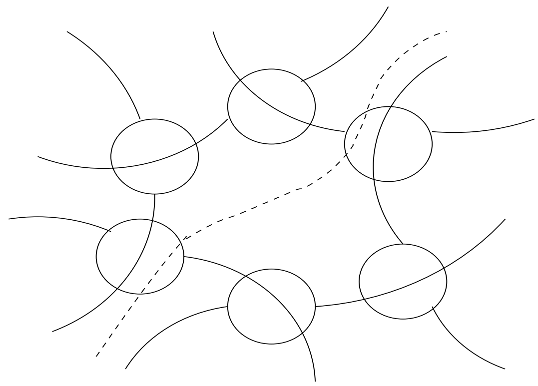

Figure 2: Two successive bubble crossings.

Lemma 2.2 *The surface-line complement M is atoroidal in case it is prime. That is, M does not contain an incompressible torus.*

Proof: If T does not intersect a bubble then it is entirely contained in $\hat{E}_+$ or $\hat{E}_-$. Both are homeomorphic to $\hat{E}$, which is atoroidal since $\pi_1(\hat{E})$ is free of rank $2g \geq 2$, where g is the genus of E, and therefore does not contain $\mathbb{Z} \oplus \mathbb{Z}$. Hence T is compressible.

Suppose that T does intersect a bubble. We may assume each intersection with a bubble is a saddle shaped disc σ (see Figure 1). Consider $\mathcal{C}_+ = E_+ \cap T$ and $\mathcal{C}_- = E_- \cap T$. We assume that T is in general position so that both intersections are finite sets of disjoint circles on T. Again we assume that the circles are essential in the sense that their total number can not be reduced by changing T up to isotopy. We claim that $\mathcal{C}_+$ or $\mathcal{C}_-$ must contain a circle that bounds a disc in T. For suppose that $\mathcal{C}_+$ does not. Then all curves in $\mathcal{C}_+$ are parallel in T and $T - \mathcal{C}_+$ is a finite set of annuli. Note that if σ is a saddle shaped disc in T then σ must be contained in one of these annuli with its positive $\partial\sigma$-arcs on the boundary of the annulus. Let A be one of the annuli in $T - \mathcal{C}_+$ and $\sigma_1,...,\sigma_k$ be the saddle shaped discs contained in the closure $\bar{A}$ of A. Then at least one of the components of $\bar{A} - \{\sigma_1, ...\sigma_k\}$ is a discs and the boundary of the closure of this disc is a curve in $\mathcal{C}_-$. See Figure 3.

Without loss of generality we may assume that C is a circle in $\mathcal{C}_+$ that bounds a disc D' in T that does not contain a curve from $\mathcal{C}_+ \cup \mathcal{C}_-$ in its interior. Since C does not intersect L it has to intersect two successive bubbles and

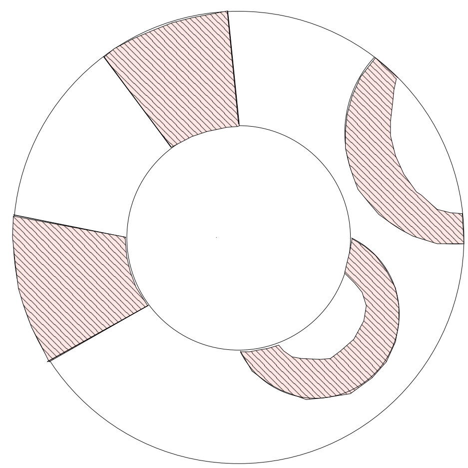

Figure 3: An annulus with saddles.

it follows from Lemma 2.2 that T also contains a meridian of L. The torus T can now be meridionally surgered. We arrive at a sphere S embedded in $\hat{E}$ that is intersected by the line L in exactly two points. Let B be the ball S bounds in $\hat{E}$. Since we assumed M to be prime it follows that the arc $B \cap L$ is unknotted in B. Hence T bounds a full torus in M which makes T compressible. •

We next show that the projection $\pi(L)$ does detect primeness of the surface-line complement M.

Lemma 2.3 *Let L be an embedded alternating line in $E \times [-1, 1]$ and M the associated surface-line complement. If the projection $\pi(L)$ is prime then so is M.*

Proof: We show that if M is not prime then neither is $\pi(L)$. Let $\mathcal{S}$ be the set of all spheres in $\hat{E}$ that intersect L in exactly two points in which the arc $B \cap L$ is knotted in B, where B is the ball in $\hat{E}$ bounded by S. Then $\mathcal{S}$ contains a sphere S that, up to isotopy, contains a minimal number of saddles. Let $\mathcal{C}_+ = S \cap E_+$ and $\mathcal{C}_- = S \cap E_-$. Assume C is a curve in $\mathcal{C}_+$ which is innermost, that is it bounds a disc that does not contain a curve from $\mathcal{C}_+ \cup \mathcal{C}_-$ in its interior. If C does not intersect L it has to cross two successive bubbles. Thus by Lemma 2.1 there is an annulus in S, part of which is a saddle σ, that contains a circle isotopic to a meridian of L. Thus there is a disc D in M that intersects L in one point and $\partial D \subseteq S$. Now the circle ∂D divides S into to discs D_1 and D_2, each intersecting L in exactly one point and replacing D_i by D yields a sphere S_i that intersects L in exactly two points and, up to isotopy, does not contain the saddle σ. So both S_i contain fewer saddles than S. Let B_i be the ball in $\hat{E}$ bounded by S_i and A_i be the arc $B_i \cap L$. At least one of the arcs A_i has to be knotted in B_i, otherwise $B \cap L$ would not be knotted in the in the ball B bounding S. Thus one of the S_i is contained in $\mathcal{S}$, contradicting the minimal number of saddles assumption on S. It follows that the curve C intersects L. Since we assumed S to intersect L in exactly two points, say p and q, it follows that C intersects L in exactly one point, say p, or in both points p and q.

Let us first consider the case $C \cap L = \{p, q\}$. Assume C does intersect a bubble. We know that C can not intersect intersects two successive bubbles because by Lemma 2.1 S would have to contain a meridian and surgery along it would produce a sphere $\mathcal{S}$ that contains fewer saddles. Thus C intersect exactly one bubble H. Isotoping S if needed we may assume that the intersection of C with the bubble consists of the two positive $\partial\sigma$ arcs, where σ is a saddle in the bubble H. We are now in the situation as shown on the right of Figure 4. Clearly S contains a meridian which can not happen under the minimal saddle assumption. It follows that C does not intersect a bubble. Since C bounds a disc in the sphere S it bounds a disc D in E_+ (incompressibility of E_+). This disc must contain a bubble (i.e. a crossing), otherwise $B \cap L$ would not be knotted in B. The projection $\pi : \hat{E} \to E$ sends the disc D to a disc that contains a crossing and whose boundary intersects $\pi(L)$ in exactly two points. Hence $\pi(L)$ is not prime.

Next we consider the case $C \cap L = \{p\}$. Now C has to cross a bubble and since it can not cross two consecutive bubbles (minimality assumption on the number of saddles) it crosses only one bubble once along a positive $\partial\sigma$ arc of some saddle σ. The situation is depicted on the left of Figure 4. Now C bounds a disc D in S whose interior lies in $\hat{E}_+$. We can push the disc $\sigma \cup D$ into $\hat{E}_-$ and eliminate the saddle σ without introducing new saddles. This contradicts the choice of S. •

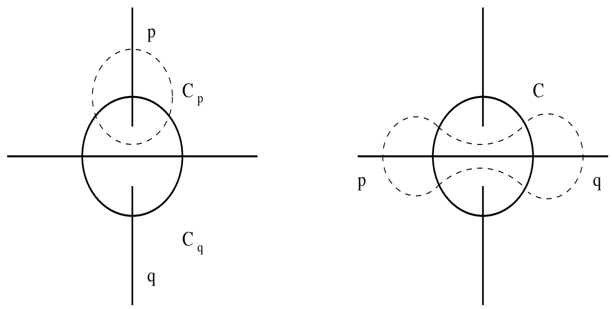

Figure 4: Circles in $\mathcal{C}_+$ that intersect L.

3 Geometry

Theorem 3.1 *(Thurston's hyperbolization). Let $\bar{M}$ be a compact, orientable, irreducible and atoroidal 3-manifold. If $\pi_1(\bar{M})$ is not virtually abelian, and $\bar{M}$ contains a properly embedded incompressible surface (i.e. $\bar{M}$ is Haken), then the interior M of $\bar{M}$ admits a complete hyperbolic structure.*

For the proof of this theorem we refer to Thurston [7], [8], [9], [10], or Kapovich [5].

Let $\bar{M}$ be a surface-arc complement as in Theorem 1.1. Then $\bar{M}$ is compact, orientable, irreducible and, by the result in the previous section, atoroidal. As a manifold with incompressible boundary, $\bar{M}$ is Haken. Note that if we attach a meridional disc to $\bar{M}$ we obtain a space homotopically equivalent to $F \times [-1, 1]$. In particular $\pi_1(F)$, which is free of rank $2g \geq 2$, is an image of $\pi_1(\bar{M})$. Thus $\pi_1(\bar{M})$ is not virtually abelian. So $\bar{M}$ satisfies all the conditions required for hyperbolization. Moreover, since no boundary component of $\bar{M}$ is a torus ($\partial\bar{M}$ is a surface of genus $2g + 1 \geq 3$), the hyperbolic structure may be taken to be convex co-compact (see Morgan [6] and also Kapovich [5]). In particular $\pi_1(\bar{M})$ is a δ-hyperbolic group. This completes the proof of Theorem 1.1.

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