

## THE $\Sigma^3$ -CONJECTURE FOR METABELIAN GROUPS

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### ABSTRACT

The  $\Sigma^3$ -conjecture for metabelian groups is proved in the split extension case.

### 1. Introduction

A metabelian group  $G$  is an extension of abelian groups  $M$  and  $Q$  ( $M$  being the kernel and  $Q$  the quotient).  $G$  is uniquely given, up to isomorphism, by the abelian group  $Q$ , the  $Q$ -module  $M$  (by conjugation in  $G$ ), and the cohomology class of the extension in  $H^2(Q, M)$ . For finitely generated metabelian groups, Bieri and Strebel [11] defined a certain subset  $\Sigma(M)$  of the sphere  $S^{n-1}$ , where  $n$  is the torsion-free rank of  $Q$ , that was shown to contain complete information about the finite presentability of  $G$ . Since then this geometric invariant has undergone quite an evolution. Geometric invariants  $\Sigma^m(G)$  can be defined for any group of type  $F_m$  and have been studied extensively for various types of group. Recently the theory of geometric invariants has been further generalised by the introduction of invariants associated with groups acting on non-positively curved spaces [3–5]. Nevertheless, two fundamental conjectures about metabelian groups remain unsettled, the  $FP_m$ -conjecture and the  $\Sigma^m$ -conjecture. The  $FP_m$ -conjecture states that  $\Sigma^1(G)$  contains complete information about the higher finiteness properties of the metabelian group  $G$ , and not just finite presentability. The  $\Sigma^m$ -conjecture, loosely formulated, states that the complement of  $\Sigma^m(G)$  is obtained from the complement of  $\Sigma^1(G)$  by a simple process involving the taking of convex hulls.

This paper is concerned with the  $\Sigma^3$ -conjecture. We show that the  $\Sigma^3$ -conjecture holds in the split case, where  $G$  is the split-extension of the  $Q$ -module  $M$  by  $Q$ . The validity of the  $\Sigma^2$ -conjecture in the split case has been established already by the second author in [17], and recently the general non-split case has been proved in [14].

We also shed some light on the question of how the two conjectures are related. In fact, the method of proof presented here is modeled on the techniques developed in paper [8] by Bieri and the first author and the idea of the diameter of cells introduced in [17]. The main point of [8] was the construction of a classifying  $K(M, 1)$ -complex for the  $Q$ -module  $M$  with  $Q$ -finite 3-skeleton (under the assumption that  $M$  is 3-tame), whereas here we construct a classifying  $K(M, 1)$ -complex for the  $Q_\chi$ -module  $M$ , where  $\chi: Q \rightarrow \mathbb{R}$  is a non-trivial (that is, non-zero) character and  $Q_\chi$  is the positive submonoid of  $Q$  with respect to  $\chi$ . In a sense one could say that the  $\Sigma^m$ -conjecture is a monoid version of the  $FP_m$ -conjecture.

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Received 10 May 2001; revised 22 March 2002.

2000 *Mathematics Subject Classification* 20F05, 20F16, 20F32.

The second author was partially supported by CNPQ, Brazil.

In the following we give precise definitions, statements of the conjectures and our results. Suppose that  $G$  is a finitely generated group. Then the set

$$S(G) = \{[\chi] = \mathbb{R}_{>0}\chi \mid \chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\}\}$$

is called the character sphere of  $G$ . Note that  $\text{Hom}(G, \mathbb{R})$  is an  $n$ -dimensional real vector space, where  $n$  is the torsion-free rank of the abelianization of  $G$ . More precisely there is an isomorphism between  $\mathbb{R}^n$  and  $\text{Hom}(G, \mathbb{R})$  via

$$u \longmapsto \chi_u = (u, -),$$

where  $(, )$  denotes the standard inner product on  $\mathbb{R}^n$ . This sets up a bijection between the unit sphere  $S^{n-1}$  in  $\mathbb{R}^n$  and the character sphere of  $G$ .

Let  $M$  be a (left)  $\mathbb{Z}[G]$ -module. Then the homological  $\Sigma^m$ -invariant is defined by Bieri and Renz [10] to be

$$\Sigma^m(G, M) = \{[\chi] \mid M \text{ is of homological type } FP_m \text{ over } \mathbb{Z}[G_\chi]\} \subseteq S(G),$$

where  $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$ . The invariants  $\Sigma^m(G)$  are mentioned in [10, Chapter 6] and should be thought of as the homotopical version of the invariants  $\Sigma^m(G, \mathbb{Z})$ . More details about  $\Sigma^m(G)$  can be found in the introduction of [21].

If  $S$  is a subset of  $\mathbb{R}^n$ , then we denote by  $\text{conv}(S)$  the convex hull of  $S$  and by  $\text{conv}_{\leq m}(S)$  we denote  $\bigcup \text{conv}(C)$ , where the union is taken over all subsets  $C$  of  $S$  containing no more than  $m$  points.

*FP<sub>m</sub>-CONJECTURE* [6]. Suppose that  $G$  is a finitely generated metabelian group. Then  $G$  is of homological type  $FP_m$  if and only if

$$0 \notin \text{conv}_{\leq m}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

Here  $\Sigma^m(G)^c$  is the complement of  $\Sigma^m(G)$  in  $S(G)$  and

$$\mathbb{R}_{>0}\Sigma^m(G)^c = \{\chi \in \text{Hom}(G, \mathbb{R}) \setminus \{0\} \mid [\chi] \in \Sigma^m(G)^c\} \subseteq \text{Hom}(G, \mathbb{R}) \simeq \mathbb{R}^n.$$

*Σ<sup>m</sup>-CONJECTURE* (Bieri). Suppose that  $G$  is a metabelian group of homological type  $FP_m$ . Then

$$\mathbb{R}_{>0}\Sigma^m(G)^c = \mathbb{R}_{>0}\Sigma^m(G, \mathbb{Z})^c = \text{conv}_{\leq m}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

Let us review the status of these conjectures. The  $FP_m$ -conjecture is true for  $m = 2$  [11], for  $m = 3$  in the split case [8], for  $G$  of finite Prüfer rank [1], and for the case when  $M$  is torsion and of Krull dimension 1 [13, 15]. The direction ( $\Rightarrow$ ) of the  $FP_m$ -conjecture is true in the split case [15, 16] (see also [22] for  $M$  torsion-free), and in the case when  $M$  is torsion [15, 16].

The  $\Sigma^m$ -conjecture is known to hold for  $m = 2$  (see [14, 17]), for  $G$  of finite Prüfer rank [20, 21], and for the case when  $M$  is torsion and of Krull dimension 1 [18]. Furthermore, the inclusion

$$\text{conv}_{\leq m}(\mathbb{R}_{>0}\Sigma^1(G)^c) \subseteq \mathbb{R}_{>0}\Sigma^m(G)^c$$

has been established [18] for metabelian groups for which the extension splits or  $M$  is torsion.

In this paper we obtain the  $\Sigma^3$ -conjecture in the split extension case as a corollary of the following.

**THEOREM 1.1.** *Let  $G$  be a split extension of abelian groups  $M$  and  $Q$  that is of type  $FP_3$ . Suppose that  $\chi$  is a real non-trivial character of  $G$  such that  $\chi(M) = 0$  and*

$$\chi \notin \text{conv}_{\leq 3}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

*Then  $[\chi] \in \Sigma^3(G, \mathbb{Z})$ .*

**COROLLARY 1.2.** *The  $\Sigma^3$ -conjecture holds true in the split case.*

Let us show how the corollary follows from Theorem 1.1. First we remark that  $\Sigma^m(G) = \Sigma^m(G, \mathbb{Z})$  for split-metabelian groups of type  $F_m$ . This follows from the validity of the  $\Sigma^2$ -conjecture established in [17] for  $m = 2$  and for  $m \geq 3$  from the general Hurewicz type formula  $\Sigma^m(G, \mathbb{Z}) \cap \Sigma^2(G) = \Sigma^m(G)$ . Thus all we have left to show is

$$\mathbb{R}_{>0}\Sigma^3(G, \mathbb{Z})^c = \text{conv}_{\leq 3}(\mathbb{R}_{>0}\Sigma^1(G)^c).$$

Since the direction  $\supseteq$  is known from previous work [18], we are left with  $\subseteq$ . Therefore suppose that  $[\chi] \notin \text{conv}_{\leq 3}(\mathbb{R}_{>0}\Sigma^1(G)^c)$ . If  $\chi(M) \neq 0$ , then it is known [19, Theorem C] that  $[\chi] \in \Sigma^m(G)$  for all  $m \geq 1$ . If  $\chi(M) = 0$ , then Theorem 1.1 tells us that  $[\chi] \in \Sigma^3(G, \mathbb{Z})$ .

## 2. Geometric properties of $\Sigma^0$ and $\Sigma^1$

The geometric structure of  $\Sigma^1(G)$  for general finitely generated groups  $G$  is unknown except that it is an open subset of  $S(G)$ . In fact for every  $m$  the geometric invariants  $\Sigma^m(G, A)$  and  $\Sigma^m(G)$  are open in  $S(G)$  [10]. In the case when  $G$  is metabelian,  $\Sigma^1(G)^c$  is a rationally defined spherical polyhedron, that is, a finite union of finite intersections of closed half subspheres, where every subsphere is defined by a rational point in the unit sphere  $S(G)$  [7]. In particular, the rational points in  $\Sigma^1(G)^c$  form a dense subset. It is interesting to note that there are examples of subgroups  $G$  of the group of the PL homeomorphisms of the interval  $[0, 1]$  for which  $\Sigma^1(G)^c$  has only one or two points, neither of which is necessarily rational [12, Theorem 8.1].

In the rest of this section we consider the invariant  $\Sigma_M(Q) = \Sigma^0(Q, M)$ , where  $Q$  is a free abelian group of rank  $n$  and  $M$  is a finitely generated (left)  $\mathbb{Z}[Q]$ -module. We view  $Q$  as the lattice  $\mathbb{Z}^n$  in the Euclidean space  $\mathbb{R}^n$  equipped with the standard inner product  $(\cdot, \cdot)$  and the corresponding norm  $|\cdot|$ .  $\text{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$  is identified with  $\mathbb{R}^n$ , where  $v \in \mathbb{R}^n$  corresponds to the homomorphism sending an element  $q$  from  $Q$  to  $(q, v)$ . For an extension  $G$  of  $M$  by  $Q$ , the map  $\pi_* : S(Q) \rightarrow S(G)$  induced by the projection  $\pi : G \rightarrow Q$  gives a bijection between  $\Sigma_M^c(Q) = S(Q) \setminus \Sigma_M(Q)$  and  $\Sigma^1(G)^c$ . Although it is not obvious from the definition,  $\Sigma_M(Q)$  does not depend on the whole structure of  $M$  as a  $\mathbb{Z}[Q]$ -module but only on the annihilator of  $M$  in  $\mathbb{Z}[Q]$  [11, Proposition 2.1]. This fact was recently refined in [9], where it was shown that a finite subset of the centralizer of  $M$  in  $\mathbb{Z}[Q]$  determines  $\Sigma_M(Q)$ .

**PROPOSITION 2.1 [9].** *Suppose that  $M$  is a finitely generated  $\mathbb{Z}Q$ -module. Then there exists a finite subset  $\Lambda$  of the centraliser of  $M$  in  $\mathbb{Z}Q$  and some  $v > 0$  such that for every  $[\mu] \in \Sigma_M(Q)$ , there is an element  $\lambda$  in  $\Lambda$  with*

$$\min\{\mu(q) \mid q \in \text{supp } \lambda\} > v.$$

COROLLARY 2.2 [11, Lemma 1.1]. *Let  $\Lambda$  be the set and  $v$  be the positive real number given by Proposition 2.1. Then there is a positive integer  $\rho_1(v)$  such that for  $x \in \mathbb{R}^n$  with  $|x| \geq \rho_1(v)$ ,  $x/|x| \in -\Sigma_M(Q)$ , there is a  $\lambda \in \Lambda$  such that  $x + \text{supp } \lambda$  is a subset of the open ball with centre the origin and radius  $|x| - v/2$ .*

We finish this section with a corollary that follows immediately from the polyhedral structure of  $\Sigma^1(G)^c$  and the fact that for split extension metabelian groups  $G$  of type  $FP_m$ , the trivial character is not in  $\text{conv}_{\leq m}(\mathbb{R}_{>0}\Sigma^1(G)^c)$  [15].

COROLLARY 2.3. *Suppose that  $M$  is a finitely generated  $\mathbb{Z}[Q]$ -module,  $\chi$  is a non-trivial real character of  $Q$  and*

$$\{0, \chi\} \cap \text{conv}_{\leq m}(\mathbb{R}_{>0}\Sigma_M(Q)^c) = \emptyset.$$

*Then there exists a positive real number  $\epsilon$  depending on  $\chi$  with the following property. Suppose that  $(x_1, x_2, \dots, x_m)$  is an  $m$ -tuple of non-trivial real characters of  $Q$  such that, for some  $m$ -tuple  $(\mu_1, \dots, \mu_m)$  of non-trivial real characters of  $Q$  with  $[\mu_i] \in \Sigma_M^c(Q)$ , the angle between  $x_i$  and  $\mu_i$  is at most  $\epsilon$  for every  $i \leq m$ . Then  $x_1, \dots, x_m, -\chi$  lie in an open half subspace of  $\mathbb{R}^n$ .*

### 3. Constructing a complex for $M$

A standard  $(Q - K(M, 1))$ -complex is a  $K(M, 1)$ -complex with a single 0-cell with a  $Q$ -action that is free on cells of dimension greater than zero and that makes the fundamental group into a  $Q$ -module isomorphic to  $M$ . Similarly, we define a standard  $(Q_\chi - K(M, 1))$ -complex substituting, in the above definition,  $Q$  by  $Q_\chi$ .

THEOREM 3.1. *Suppose that  $X$  is a standard  $(Q_\chi - K(M, 1))$ -complex with  $Q_\chi$ -finite  $k$ -skeleton. Then  $[\chi]$  is contained in  $\Sigma^k(G)$ .*

*Proof.* Let  $\tilde{X}$  be the universal covering of  $X$  and let  $\tilde{*}$  be a vertex that maps to the unique vertex  $*$  of  $X$  under the covering projection. The  $Q_\chi$ -action on  $X$  lifts uniquely to a  $Q_\chi$ -action on  $\tilde{X}$ , fixing the vertex  $\tilde{*}$ , and one uses the  $M$ -action on  $\tilde{X}$  by covering transformations to construct a  $G_\chi$ -action, where  $G_\chi$  is the split extension of  $M$  by  $Q_\chi$ . If  $\tilde{x}$  is a point in  $\tilde{X}$  and  $(m, q) \in G_\chi$ , then simply define

$$(m, q)\tilde{x} = m(q\tilde{x}).$$

One can check that this makes  $\tilde{X}$  into a  $G_\chi$ -complex with vertex stabilizers all isomorphic to  $Q_\chi$  and trivial stabilizers of higher-dimensional cells. If  $k = 2$ , then we consider an obvious  $Q$ -extension  $X_0$  of  $\tilde{X}$  on which  $G = M \rtimes Q$  acts freely on all edges and 2-cells, the vertex stabilizers are isomorphic to  $Q$ , and the action is cocompact. Then, as  $\tilde{X}$  is simply connected, the main result of [21] implies the theorem.

In general, for  $k \geq 3$ , we use the formula  $\Sigma^k(G) = \Sigma^k(G, \mathbb{Z}) \cap \Sigma^2(G)$  and reduce the theorem to  $[\chi] \in \Sigma^k(G, \mathbb{Z})$ . Now we consider the cellular chain complex of  $\tilde{X}$

$$\mathcal{F} : F_k \longrightarrow \dots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow \mathbb{Z}[M] \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and note that it has the property that all  $F_i$  for  $i \leq k$  are free  $\mathbb{Z}[G_\chi]$ -modules of finite rank. In particular, the augmentation ideal  $\text{Aug } \mathbb{Z}[M]$  is of homological type  $FP_{k-1}$  over  $\mathbb{Z}[G_\chi]$ , where  $M$  acts on the augmentation ideal via multiplication and

$Q_\chi$  via its action on  $M$ , that is,  $[\chi] \in \Sigma^{k-1}(G, \text{Aug } \mathbb{Z}[M])$ . The result now follows from the following.

LEMMA 3.2. For  $k \geq 1$ ,

$$\Sigma^k(G, \mathbb{Z}) \cap \{[\chi] \mid \chi(M) = 0\} = \Sigma^{k-1}(G, \text{Aug } \mathbb{Z}[M]) \cap \{[\chi] \mid \chi(M) = 0\}.$$

*Proof.* In this proof we consider only real characters  $\chi$  of  $G$  such that  $\chi(M) = 0$ . By definition,  $[\chi] \in \Sigma^k(G, \mathbb{Z})$  if and only if  $\mathbb{Z}$  is of type  $FP_k$  over  $\mathbb{Z}[G_\chi]$ . Using the dimension shifting argument [2, Proposition 1.4] for the short exact sequence

$$0 \rightarrow \text{Aug } \mathbb{Z}[G_\chi] \rightarrow \mathbb{Z}[G_\chi] \rightarrow \mathbb{Z} \rightarrow 0,$$

we see that  $\mathbb{Z}$  is  $FP_k$  over  $\mathbb{Z}[G_\chi]$  if and only if  $\text{Aug } \mathbb{Z}[G_\chi]$  is of type  $FP_{k-1}$  over  $\mathbb{Z}[G_\chi]$ .

Note that since  $Q$  is abelian,

$$\Sigma^\infty(Q, A) = \bigcap_{j \geq 1} \Sigma^j(Q, A) = \Sigma^0(Q, A)$$

for every finitely generated  $\mathbb{Z}[Q]$ -module  $A$ . In particular,  $S(Q) = \Sigma^\infty(Q, \mathbb{Z})$  and  $\mathbb{Z}$  is of type  $FP_\infty$  over  $\mathbb{Z}[Q_\chi]$ , although  $\mathbb{Z}[Q_\chi]$  is not necessarily Noetherian. By the dimension shifting argument for the short exact sequence

$$0 \rightarrow \text{Aug } \mathbb{Z}[Q_\chi] \rightarrow \mathbb{Z}[Q_\chi] \rightarrow \mathbb{Z} \rightarrow 0,$$

the augmentation ideal  $\text{Aug } \mathbb{Z}[Q_\chi]$  is of type  $FP_\infty$  over  $\mathbb{Z}[Q_\chi]$  and then the induced module

$$\begin{array}{c} \mathbb{Z}[G_\chi] \uparrow \\ \mathbb{Z}[Q_\chi] \uparrow \end{array} \text{Aug } \mathbb{Z}[Q_\chi]$$

is  $FP_\infty$  over  $\mathbb{Z}[G_\chi]$ . Consider the short exact sequence

$$0 \rightarrow \begin{array}{c} \mathbb{Z}[G_\chi] \uparrow \\ \mathbb{Z}[Q_\chi] \uparrow \end{array} \text{Aug } \mathbb{Z}[Q_\chi] \simeq \mathbb{Z}[G_\chi] \cdot \text{Aug } \mathbb{Z}[Q_\chi] \xrightarrow{\tau} \text{Aug } \mathbb{Z}[G_\chi] \rightarrow \text{Aug } \mathbb{Z}[M] \rightarrow 0,$$

where  $\tau$  is the inclusion map. By the dimension shifting argument,  $\text{Aug } \mathbb{Z}[G_\chi]$  is  $FP_{k-1}$  over  $\mathbb{Z}[G_\chi]$  if and only if  $\text{Aug } \mathbb{Z}[M]$  is  $FP_{k-1}$  over  $\mathbb{Z}[G_\chi]$ , that is,  $[\chi] \in \Sigma^{k-1}(G, \text{Aug } \mathbb{Z}[M])$ . This completes the proof.  $\square$

We will next construct the 3-skeleton  $X$  of a  $(Q_\chi - K(M, 1))$ -complex in almost complete analogy to the construction of a

$$(Q - K(M, 1)^3)\text{-complex}$$

given in [8].

Since  $[\chi]$  is assumed to be in  $\Sigma_M(Q)$ , we can choose a resolution of the  $Q_\chi$ -module  $M$  by finitely generated  $Q_\chi$ -modules

$$\mathbb{Z}B \xrightarrow{\hat{\varrho}} \mathbb{Z}A \rightarrow M \rightarrow 0,$$

where  $A$  and  $B$  are free  $Q_\chi$ -sets with finitely many orbits. Fix sets of orbit representatives  $A_0, B_0$  for  $A, B$ , respectively.

It is convenient to put an ordering on  $A$  and  $B$  that is compatible with the  $Q_\chi$ -action. Such an ordering can be obtained from an ordering of  $Q_\chi$  and an ordering on the fixed finite sets  $A_0, B_0$  of orbit representatives.

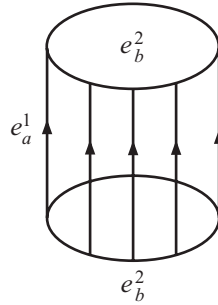


FIGURE 1. Prism 3-cell.

Let  $Z$  be the subset of the product of spheres  $\prod_{a \in A} S_a^1$  consisting of all tori  $S_{a_1}^1 \times \dots \times S_{a_k}^1$  with  $a_1 < \dots < a_k$ . Now  $Q_\chi$  acts in the obvious way on the product of spheres and makes  $Z$  into a cubical  $Q_\chi$ -complex. Note that  $Z$  is a  $K(\mathbb{Z}A, 1)$ -complex with a single vertex  $*$  and that the 1-cells  $e_a^1$  of  $Z$  are in one-to-one correspondence with the elements of  $A$ . We write  $e_{a_1, a_2}^2$  for  $e_{a_1}^1 e_{a_2}^1$ . For every  $b \in B$  we attach to  $Z$  a 2-cell  $e_b^2$  as follows. If

$$\partial(b) = \sum_{i=1}^k n_i a_i,$$

where  $a_1 < \dots < a_k$ , use the attaching path  $w = e_{a_1}^{n_1} \dots e_{a_k}^{n_k}$  (we use the ordering on  $A$  to obtain a uniquely defined edge path from a sum on the  $a_i$ ).  $Q_\chi$  acts on the set of new 2-cells by  $qe_b^2 = e_{qb}^2$ .

For every pair  $(b, a)$  we furthermore attach a 3-cell  $e_{a,b}^3$ , as shown in Figure 1. This 3-cell is the product of the 2-cell  $e_b^2$  and the 1-cell  $a$ . We refer to such 3-cells as prisms.

Again,  $Q_\chi$  acts on this set of 3-cells by acting on the labels  $qe_{a,b}^3 = e_{qa, qb}^3$ . We call the resulting  $Q_\chi$ -complex  $Y$ . The action induces  $Q_\chi$ -module structures on all homotopy groups and turns the fundamental group into a  $Q_\chi$ -module isomorphic to  $M$ .

LEMMA 3.3. *The second homotopy module  $\pi_2(Y)$  is a finitely generated  $Q_\chi$ -module.*

*Proof.* Consider the composition of  $Q_\chi$ -module maps

$$\pi_2(Y) \xrightarrow{h} H_2(Y) \xrightarrow{j} H_2(Y, Z) = \mathbb{Z}B \xrightarrow{\partial} H_1(Z) = \mathbb{Z}A,$$

where  $h$  is the Hurewicz map and  $j$  is induced by inclusion. One shows exactly as in [8] (by combinatorial homotopy arguments) that the composition  $j \circ h$  is injective. We next show that the image of this composition is the kernel of the boundary map  $\partial : \mathbb{Z}B \rightarrow \mathbb{Z}A$ . Since  $j(H_2(Y)) = \ker(\partial)$ , it is clear that  $j \circ h(\pi_2(Y)) \subseteq \ker(\partial)$ . Let us show the other inclusion. Suppose that  $\beta = \sum n_i b_i$  is an element in  $\ker(\partial)$ . Then, by construction of the cells  $e_b^2$ , we see that the boundary  $w$  of the union (planar diagram)

$$d_1 = \bigcup n_i e_{b_i}^2$$

(see Figure 2) represents the trivial element of  $H_1(Z) = \pi_1(Z) = \mathbb{Z}A$ . Thus there exists a contracting disc  $d_2$  in  $Z$  with  $\partial(d_2) = \partial(d_1)$ . Now the union  $d_1 \cup -d_2$  makes a sphere that represents an element  $\alpha$  of  $\pi_2(Y)$  such that  $j \circ h(\alpha) = \beta$ .

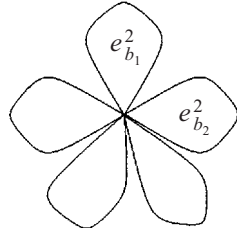


FIGURE 2. Planar diagram  $d_1$ .

Since  $M$  is finitely generated as a  $Q_\chi$ -module (we assumed that  $[\chi]$  was in  $\Sigma_M(Q)$ ), it is actually of type  $FP_\infty$ . In particular, the kernel of  $\partial$  is a finitely generated  $Q_\chi$ -module.  $\square$

Lemma 3.3 shows that we can attach finitely many  $Q_\chi$ -orbits of 3-cells to  $Y$  to obtain the 3-skeleton  $X$  of a  $(Q_\chi - K(M, 1))$ -complex.

#### 4. Support geometry and pushing cells

Define the support of the 0-cell  $*$  in  $X$  to be the subset of  $Q$  consisting of the trivial element. The support of a 1-cell  $e_a^1$  of  $X$ ,  $a = qa_0$ ,  $a_0 \in A_0$ ,  $q \in Q$  is defined to be  $\text{supp}(e_a^1) = \{q\}$ . If  $c$  is a higher-dimensional cell of  $X$ , then we define

$$\text{supp}(c) = \{q \in Q \mid q \text{ is in the support of a 1-cell occurring in } \partial(c)\}.$$

In [8], the diameter of a cell was defined to be the minimal radius of a ball in  $\mathbb{R}^n$  that contains the support of the cell. However, this notion is not suitable for working over the halfspace  $\mathbb{R}_\chi^n = \{r \in \mathbb{R}^n \mid \chi(r) \geq 0\}$ .

Let  $I(m, z_0)$  be a cube with edge length  $2m$  and centre  $(m + z_0)[\chi]$ , where  $z_0$  is a fixed positive real number, and is such that one of its walls is perpendicular to  $[\chi]$ . At this stage we do not pose any restrictions on  $z_0$ , but later on we will encounter technical points that can be resolved only when  $z_0$  is sufficiently large. Now let  $B(m, z_0)$  be the union of all closed balls with radius  $z_0$  and center in  $I(m, z_0)$  (see Figure 3).

We define the *global diameter* of a cell  $c$  in  $X$  to be

$$\min\{m \in \mathbb{R} \mid \text{supp}(c) \text{ is a subset of some } \mathbb{R}_\chi^n\text{-translate of } B(m, z_0)\}.$$

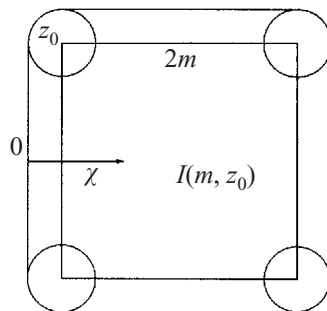


FIGURE 3.  $B(m, z_0)$ .

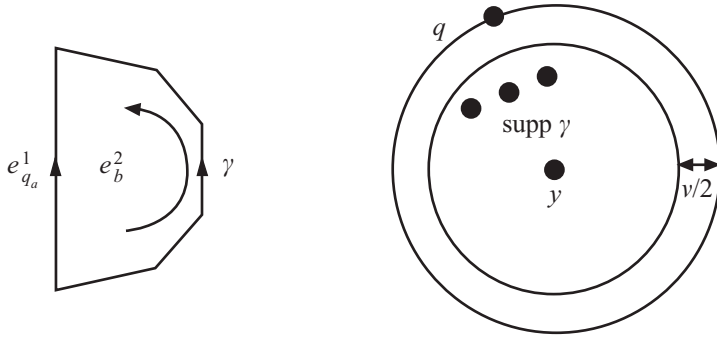


FIGURE 4. Pushing 2-cell and its support.

Note that since every  $B(m, z_0)$  can only contain finitely many integral points, the set of diameters of cells of  $X$  is a discrete subset of  $\mathbb{R}_{\geq 0}$ .

The construction of  $X$  depends on the choice of the free  $Q_\lambda$ -sets  $A$  and  $B$ . Here we impose some restrictions on set  $B$ . Fix some positive real number  $\epsilon$  given by Corollary 2.3 for  $m = 3$ ; the positive real number  $v$  is given by Proposition 2.1 and  $\rho_1(v)$  is given by Corollary 2.2. According to Proposition 2.1, there is a special finite subset  $\Lambda$  of the centralizer of  $M$  in  $\mathbb{Z}[Q]$ . For every  $\lambda \in \Lambda$  and every  $\mu \in \Sigma_M(Q)$  there is an element  $q_\lambda$  from  $\text{supp } \lambda \cup \{1\}$  such that

$$\min\{\mu(Q) \mid q \in \text{supp } \lambda \cup \{1\}\} = \mu(q_\lambda).$$

Note that  $q_\lambda^{-1} \in \mathbb{Z}Q_\lambda$ . Since  $q_\lambda^{-1}(1-\lambda)a$  is sent to the trivial element under  $\mathbb{Z}A \rightarrow M$ , we may assume that for every  $(\lambda, a) \in \Lambda \times A_0$  there exists a  $b_{\lambda,a} \in B$  such that  $\partial(b_{\lambda,a}) = q_\lambda^{-1}(1-\lambda)a$ .

Geometrically, this means that if  $q$  is an element of  $Q_\lambda$ ,  $y$  is a point of  $\mathbb{R}_\lambda^n$  such that the closed ball with centre  $y$  and radius  $|q - y| = z_0 \geq \rho_1(v)$  is inside  $\mathbb{R}_\lambda^n$  and  $(q - y)/(|q - y|) \in -\Sigma_M(Q)$ , then there is a 2-dimensional cell  $e_b^2$  in  $X$  that gives a homotopy between the edge  $e_{q_a}^1$  and a path  $\gamma$  with support  $q \text{supp } (\lambda)$  for some  $\lambda \in \Lambda$  given by Corollary 2.2 (see Figure 4). In particular, the support of the path  $\gamma$  is in the open ball with centre  $y$  and radius  $|q - y| - v/2$ . We call these special cells ‘pushing 2-cells’. Note that we have already imposed a restriction on  $z_0$  by assuming that  $z_0 \geq \rho_1(v)$ .

Let us next construct 3-dimensional pushing cells. For this it is convenient to consider a  $Q$ -enlargement  $\hat{X}$  of  $X$  that we define inductively as follows. Define  $\hat{X}^{(0)} = X^{(0)} = *$ . Suppose that  $\mathcal{C}_i$  is a set of  $Q_\lambda$ -orbit representatives for the  $i$ -dimensional cells in  $X$ ,  $i = 1, 2, 3$ . Suppose that  $\hat{X}^{i-1}$  is defined. For  $c \in \mathcal{C}_i$ ,  $q \in Q - Q_\lambda$ , attach a cell  $qc$  to  $\hat{X}^{i-1}$  along  $q\partial(c)$ . We extend the notion of cell support to  $\hat{X}$ . Here the support of a cell of  $\hat{X}$  is a finite subset of  $Q$ .

If  $\mu$  is a character and  $d$  is a real number, then define  $\hat{X}_{\mu \geq d}$  to be the subcomplex on cells  $c$  such that

$$\text{supp}(c) \subseteq \{q \in Q \mid \mu(Q) \geq d\}.$$

Note that  $X = \hat{X}_{\lambda \geq 0}$ . The existence of pushing 2-cells yields the following result (see [8, Lemma 6.1]).

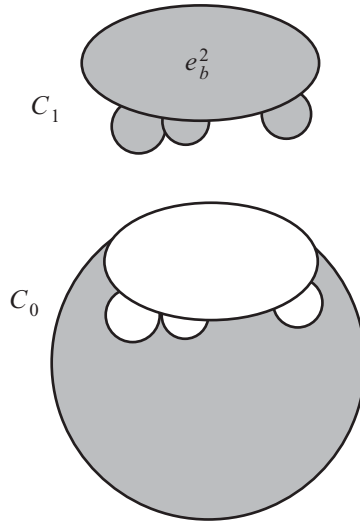


FIGURE 5. Pushing 3-cell with boundary  $-C_1 \cup C_0$ .

LEMMA 4.1. For every  $[\mu] \in \Sigma_M(Q)$  and real number  $d$ , the inclusion of  $\hat{X}_{\mu \geq d}$  in  $\hat{X}$  induces a monomorphism of  $Q_\mu$ -modules

$$\pi_1(\hat{X}_{\mu \geq d}) \longrightarrow \pi_1(\hat{X}).$$

We finish this section with the definition of some special pushing 3-cells that play an important role in the proof of the main result.

Suppose that  $w$  is the boundary path of a 2-cell  $e_b^2$ ,  $b \in B$ , of  $\hat{X}$ , with edges  $e_{a_i}^1$  with support  $q_i$ . Assume that the support  $\{q_1, \dots, q_m\}$  of  $e_b^2$  is inside  $v_0 + B(\rho, z_0)$  for some  $v_0 \in \mathbb{R}_{\chi \geq 0}^n$ , and that at least one  $q_i$  is on the boundary of  $v_0 + B(\rho, z_0)$ . We assume further that for every  $q_i$  that is on the boundary of  $v_0 + B(\rho, z_0)$ ,

$$\chi_i = \frac{q_i - y_i}{|q_i - y_i|} \in -\Sigma_M(Q),$$

where  $y_i$  is the projection of  $q_i$  to  $v_0 + I(\rho, z_0)$ , that is,  $y_i$  is the unique point from  $v_0 + I(\rho, z_0)$  such that the distance between  $y_i$  and  $q_i$  is  $z_0$ . Let  $I_w$  be the subset of  $\{1, \dots, m\}$  for which the latter holds,  $Y_w = \{y_i \mid i \in I_w\}$ , and suppose that the support of  $w$  is in  $U_w(z_0)$  (this holds if  $z_0$  is sufficiently large), where  $U_w(r)$  is the union over  $i \in I_w$  of all closed balls with centre  $y_i$  and radius  $r$ . Now, to every edge  $e_{a_i}^1$  with  $i \in I_w$ , a pushing cell  $e_{b_i}^2$  is glued with the property that  $\text{supp}(e_{b_i}^2) \setminus \{q_i\}$  is in the open ball with centre  $y_i$  and radius  $z_0$ . Let us call this union of 2-cells  $C_1$  (see top of Figure 5). Note that for our fixed  $e_b^2$ , there are only finitely many such configurations  $C_1$ . The boundary path  $\hat{w}$  of  $C_1$  has support in  $\bigcup_{r < z_0} U_w(r)$ , so is strictly inside  $v_0 + B(\rho, z_0)$ . Our aim is to show that if  $z_0$  is sufficiently large, then the path  $\hat{w}$  has a contracting disc  $C_0$  with support entirely contained in the interior of  $v_0 + B(\rho, z_0)$ .

Using the fact that  $\Sigma_M(Q)$  is an open subset of  $S(Q)$ , if  $z_0$  is sufficiently large, then for every  $q_i$  from the support of  $w$  the direction  $\mu_{q_i} = (q_i - y)/(|q_i - y|) \in -\Sigma_M(Q)$ , where  $y$  is  $(1/|I_w|) \sum_{i \in I_w} \bar{y}_i$ ,  $\bar{y}_i$  is a point with integral coordinates in the intersection of the closed ball with centre  $q_i$  and radius  $z_0$  with the closed ball

with centre  $y_i$  and radius  $\sqrt{n} \leq z_0$ , and the average direction  $\mu = \sum \mu_{q_i} / |\sum \mu_{q_i}|$  is in  $-\Sigma_M(Q)$ . By Lemma 4.1, we know that  $\hat{w}$  is contractible in  $\hat{X}_{-\mu \geq d}$ , where  $d = \min\{-\mu(Q) \mid q \in \text{supp } \hat{w}\}$ . If  $z_0$  is sufficiently large, the fact that  $\hat{w}$  is contractible in the halfspace implies that it is contractible by a disc  $C_0$  with support in  $U_w(z'_0)$ ,  $z'_0 < z_0$ , and is hence in the interior of  $v_0 + B(\rho, z_0)$ . Add a free  $Q$ -orbit of 3-dimensional cells  $Qc$  to  $\hat{X}$ , where  $c$  is the 3-cell with boundary  $-C_0 \cup C_1$  as shown in Figure 5. Note that the construction of  $C_0$  depends on  $C_1$  (or rather its boundary  $\hat{w}$ ) and the direction  $\mu$  (which depends on the choice of  $v_0 + B(\rho, z_0)$ ). However, only finitely many directions  $\mu$  can arise for our fixed  $e_b^2$ , because every  $\bar{y}_i$  is an integer lattice point in the ball of radius  $z_0$  with center  $q_i$ . Thus, for a fixed  $C_1$ , we have built only finitely many contracting discs  $C_0$ . We call these added 3-cells ‘pushing 3-cells’. They can be used to push the boundary of  $C_1$  in the  $-\mu$  direction. We proceed in this fashion for each  $e_b^2$ ,  $b \in B$  and  $v_0 + B(\rho, z_0)$ . Note that  $Q$  acts cofinitely on the  $Q$ -orbits generated by all possible  $C_1$ . In the end, we have added  $Q$ -finitely many pushing 3-cells to  $\hat{X}$ . We denote that bigger complex also by  $\hat{X}$  and the half  $\hat{X}_{\chi \geq 0}$  by  $X$ . From now on we will work with this bigger  $(Q_\chi - K(M, 1)^{(3)})$ -complex.

5. Proof of the main result

In this section we prove Theorem 1.1 by showing that  $[\chi] \in \Sigma^3(G)$ . Suppose that  $\chi : G \rightarrow \mathbb{R}$  is a character,  $\chi(M) = 0$ , and  $\chi \notin \text{conv}_{\leq 3}(\mathbb{R}_{>0}\Sigma^1(G)^c)$ . Consider the  $Q_\chi - K(M, 1)^{(3)}$   $X$  constructed before. In order to show that  $[\chi] \in \Sigma^3(G)$ , we have to reduce it down to a  $Q_\chi$ -finite one so that we can apply Theorem 3.1. For  $\rho > 0$ , let  $X_\rho$  be the subcomplex of  $X$  on cells of global diameter less than or equal to  $\rho$ . Clearly  $X_\rho$  is a  $Q_\chi$ -finite complex. The goal of this section is to show that for some  $\rho_0$  the complex  $X_{\rho_0}$  is a

$$(Q_\chi - K(M, 1)^{(3)})\text{-complex.}$$

Let  $\{\rho_0 < \rho_1 < \rho_2 < \dots\}$  be the set of diameters of cells of  $X$  bigger than or equal to some big real number  $\rho_0$  (occurring as diameter) that we will specify as we go along. We aim to construct cellular maps  $r_i : X_{\rho_i} \rightarrow X_{\rho_{i-1}}$ , which is the identity on  $X_{\rho_{i-1}}$ ,  $i = 1, 2, \dots$ . The main theorem follows quickly once these maps are constructed. Indeed, let

$$f_i = r_i \circ \dots \circ r_1 : X_{\rho_i} \rightarrow X_{\rho_0},$$

and let  $r : X \rightarrow X_{\rho_0}$  be the union of the  $f_i$ . Then  $r$  is a retraction and hence the inclusion  $X_{\rho_0} \subseteq X$  induces monomorphisms on the homotopy groups. Thus  $\pi_2(X_{\rho_0}) = 1$ . Since both  $X$  and  $X_{\rho_0}$  have the same 1-skeleton (1-cells have global diameter 0), the inclusion induces a  $Q_\chi$ -isomorphism on fundamental groups.

5.1. A geometric lemma

The construction of the maps  $r_i$  relies on the following geometric lemma.

LEMMA 5.1. *Suppose that  $M$  is a finitely generated  $Q$ -module and  $\chi$  is a real character of  $Q$  such that*

$$\{0, \chi\} \cap \text{conv}_{\leq m}(\mathbb{R}_{>0}\Sigma_M^c(Q)) = \emptyset$$

for some  $m \geq 2$ . Then there exist positive numbers  $\epsilon_0, \tilde{\rho}$  and  $\tilde{z}_0$  with the following property.

Suppose that  $Y = \{q_1, \dots, q_s\}$  is a subset of  $B(\rho, z_0)$  of global diameter  $\rho \geq \tilde{\rho}$ ,  $z_0 \geq \tilde{z}_0$ , and for every  $q_i$  from the boundary of  $B(\rho, z_0)$  define  $y_i$  to be the projection of  $q_i$  to  $I(\rho, z_0)$  (that is,  $y_i$  is the unique element from  $I(\rho, z_0)$  such that the distance between  $y_i$  and  $q_i$  is  $z_0$ ) and set  $\chi_i = q_i - y_i$ . Assume further that one of the following holds:

- (1)  $s = m$ .
- (2)  $\{q_2, \dots, q_s\}$  is the support of some element  $e_b$ .

Then, for some  $i_0$ ,

$$[\chi_{i_0}] \in -\Sigma_M(Q),$$

and for every  $[\mu] \in S(Q)$  such that the angle between  $\mu$  and  $\chi_{i_0}$  is at most  $\epsilon_0$ ,

$$[\mu] \in -\Sigma_M(Q).$$

*Proof.* (1) We consider first the case when  $s = m$ . Set  $\epsilon_0 = \epsilon$ , where  $\epsilon$  is the positive real number given by Corollary 2.3. We assume that the lemma does not hold and for every  $\chi_i$  there is a  $[\mu_i] \notin -\Sigma_M(Q)$  such that the angle between  $\mu_i$  and  $\chi_i$  is at most  $\epsilon$ . By Corollary 2.3, the set  $X$  of all  $x_i = -\chi_i$  together with  $-\chi$  lies in an open halfspace; in particular, there is a  $v \in \mathbb{R}^n$  with arbitrary small length such that

$$|x| > |v + x| \quad \text{for every } x \in X = \{x_1, \dots, x_m, -\chi\}.$$

Then, for sufficiently small positive real number  $\alpha$  and every  $q_i$  from the boundary of  $B(\rho, z_0)$ ,

$$q_i - v \in B(\rho, z_0 - \alpha/2) \subseteq B(\rho - \alpha/2, z_0)$$

and  $0 < |-\chi|^2 - |-\chi + v|^2 = 2(v, \chi) - |v|^2$ , so  $v \in \mathbb{R}_\chi^n$ . Furthermore, if  $\alpha$  is sufficiently small and the length of  $v$  is sufficiently small for all  $q_i$  that are not on the boundary of  $B(\rho, z_0)$ , we have  $q_i \in v + B(\rho - \alpha/2, z_0)$ . In particular,  $\{q_1, \dots, q_m\}$  is a subset of  $v + B(\rho - \alpha/2, z_0)$  and its global diameter is at most  $\rho - \alpha/2$ , a contradiction.

(2) Now we consider the second case and set  $\epsilon_0 < \epsilon$ , where  $\epsilon$  is the positive real number given by Corollary 2.3. Assume that the lemma does not hold and, as in the first part, for every  $\chi_i$  there is a  $[\mu_i] \notin -\Sigma_M(Q)$  such that the angle between  $\mu_i$  and  $\chi_i$  is at most  $\epsilon_0$ . Define  $I$  to be the index set of all  $q_i \in \partial B(\rho, z_0)$ ,  $i \geq 2$ . For  $z_0$  sufficiently large and  $\epsilon_0$  sufficiently small, the elements of  $\{\mu_i\}_{i \in I}$  are very close. As  $\rho$  is the diameter of the set  $\{q_1, q_2, \dots, q_s\}$  for  $\tilde{\rho}$  sufficiently big,  $q_1$  is on the boundary of  $B(\rho, z_0)$ . Then  $\chi_1$  is defined,  $\mu_1 \in -\Sigma_M^c(Q)$ , and for  $i \in I$  we have  $\text{conv}_{\leq 2}\{-\mu_1, -\mu_i\} = \text{conv}\{-\mu_1, -\mu_i\}$  is close to  $\text{conv}\{-\mu_1, -\mu_i\}_{i \in I}$ . In other words, for sufficiently large  $\tilde{\rho}, \tilde{z}_0$  and sufficiently small  $\epsilon_0$ , the fact that  $\Sigma_M^c(Q)$  is a polyhedron together with  $\{0, \chi\} \cap \text{conv}_{\leq 2}\{-\mu_1, -\mu_i\} \subseteq \{0, \chi\} \cap \text{conv}_{\leq 2}\Sigma_M^c(Q) = \emptyset$  implies that  $\{0, \chi\} \cap \text{conv}\{-\mu_1, -\mu_i\}_{i \in I} = \emptyset$  and  $\{0, \chi\} \cap \text{conv}\{-\chi_1, -\chi_i\}_{i \in I} = \emptyset$ . Then there exists a  $v \in \mathbb{R}^n$  with arbitrary small length such that

$$|x| > |v + x| \quad \text{for every } i \in I, \quad x \in X = \{-\chi_1, -\chi_i, -\chi\},$$

and we can continue exactly as in the previous case. □

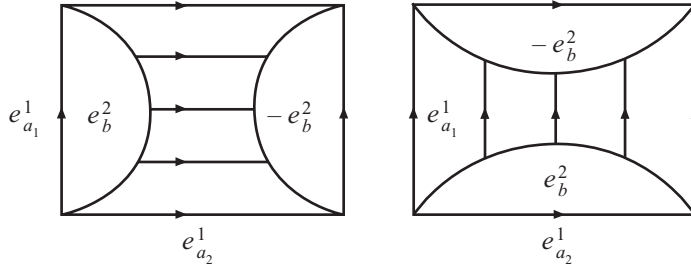


FIGURE 6. Image of a square.

5.2. Construction of the map  $r_i$

We start this subsection with the definition of  $\rho_0$ . Edges have global diameter 0 and all cells other than squares, cubes and prisms have bounded global diameter, so we can choose  $\rho_0$  big enough so that  $X_{\rho_0}$  contains all cells except big squares, big cubes and big prisms. Furthermore, we want  $\rho_0 \geq \tilde{\rho}$ , where  $\tilde{\rho}$  is given by Lemma 5.1. We define  $r_i$  on a generator of the  $Q_\chi$ -orbit of cells (of dimension 1 or more) and then extend the map in the unique way that will make it commute with the  $Q_\chi$ -action.

Let us now define the map  $r_i$ . In the following we set  $\rho = \rho_i$  and  $\rho' = \rho_{i-1}$ . On  $X_{\rho'}$  the map  $r_i$  is the identity. This implies that  $r_i$  is the identity on all edges. Let  $e^2$  be a square of global diameter  $\rho$ . Let

$$\text{supp}(e^2) = \{q_1, q_2\},$$

where  $\{q_1\}$  and  $\{q_2\}$  are the supports of the edges  $e_{a_1}^1$  and  $e_{a_2}^1$  of the square  $e^2$ . The  $q_i, i = 1, 2$ , lie on the boundary of some translate  $v_0 + B(\rho, z_0)$  for some  $v_0 \in \mathbb{R}_\chi^n$ . Let  $y_i$  be the projection of  $q_i$  in  $v_0 + I(\rho, z_0)$ . Now by the first case of Lemma 5.1 applied for  $m = 2$ , there is an  $i_0$  such that

$$\frac{q_{i_0} - y_{i_0}}{|q_{i_0} - y_{i_0}|} \in -\Sigma_M(Q).$$

Assume first that  $i_0 = 1$ . Then there is a 2-dimensional pushing cell  $e_b^2$  that gives a homotopy between the edge  $e_{a_1}^1$  and a path  $\gamma$  with support in the interior of  $v_0 + B(\rho, z_0)$ , that is, every square that is a product of an edge from  $\gamma$  with the edge  $e_{a_2}^1$  has a global diameter smaller than  $\rho$ . The image of the square is shown on the left-hand side of Figure 6. It is contained in  $X_{\rho'}$ .

We proceed in the same fashion in the case when  $i_0 = 2$ . The image of the square  $e^2$  is shown on the right-hand side of Figure 6.

It remains to extend the map

$$r_i : (X_\rho)^{(2)} \longrightarrow (X_{\rho'})^{(2)}$$

to the 3-skeleton. Define  $r_i$  to be the identity on 3-cells of diameter less than  $\rho$ . In order to show that we can extend  $r_i$  to all 3-cells  $e^3$  in  $X_\rho$ , we need to show that the boundary

$$\partial(r_i(e^3)) = r_i(\partial(e^3))$$

(which is contained in  $X_{\rho'}$ ) can be contracted in  $X_{\rho'}$ .

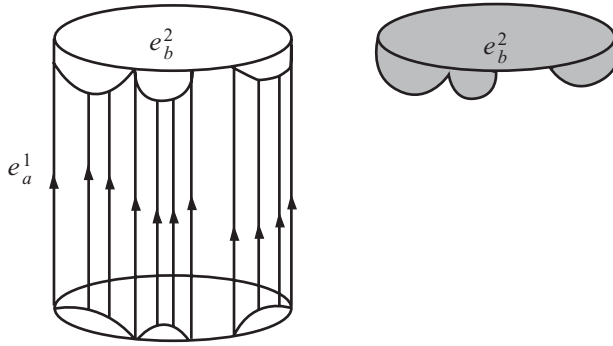


FIGURE 7.  $U_0$  and its top.

Case 1: We assume that  $e^3 = e_{a,b}^3$  is a prism of global diameter  $\rho$ ,  $\text{supp}(e_a^1) = \{q_1\}$ , and  $\text{supp}(e_b^2) = \{q_2, \dots, q_s\}$ ; hence

$$\text{supp}(e^3) = \{q_1, \dots, q_s\} \subseteq v_0 + B(\rho, z_0)$$

for some  $v_0 \in \mathbb{R}_\chi^n$ . For every  $q_i$  that is on the boundary of  $v_0 + B(\rho, z_0)$ , we define  $y_i$  to be the projection of  $q_i$  to  $v_0 + I(\rho, z_0)$ , that is,  $y_i$  is the unique element from  $v_0 + I(\rho, z_0)$  such that the distance between  $q_i$  and  $y_i$  is  $z_0$ . As there are only finitely many  $Q_\chi$ -orbits of cells  $e_b^2$ , that is, finitely many shapes for  $\text{supp}(e_b^2)$ , we see that for sufficiently large  $z_0$ , the point  $q_1$  is on the boundary of  $v_0 + B(\rho, z_0)$ , and we assume that this is the case.

Case 1.1: Suppose that  $(q_1 - y_1)/(|q_1 - y_1|) \notin -\Sigma_M(Q)$ . Let  $I_0$  be the subset of  $\{2, \dots, s-1, s\}$  of all  $i$  such that the global diameter of  $q_1, q_i$  is  $\rho$ . If  $I_0$  is not empty by the first part of Lemma 5.1 applied for  $m = 2$  and  $Y = \{q_1, q_i\}$ ,  $i \in I_0$ , then  $q_i$  has the property that  $\chi_i = (q_i - y_i)/(|y_i - q_i|) \in -\Sigma_M(Q)$ , and for every  $\mu_i \in \text{Hom}(Q, \mathbb{R}) \setminus \{0\}$  such that the angle between  $\mu_i$  and  $\chi_i$  is at most  $\epsilon_0$ ,  $[\mu_i] \in -\Sigma_M(Q)$ . Fix one  $i_0 \geq 2$  with the above property. If  $z_0$  is sufficiently large for every  $i \geq 2$  such that  $q_i$  is on the boundary of  $v_0 + B(\rho, z_0)$  but  $i \notin I_0$ , then the angle between  $\chi_i = (q_i - y_i)/(|q_i - y_i|)$  and  $\chi_{i_0}$  is at most  $\epsilon_0$ ; in particular,  $\chi_i \in -\Sigma_M(Q)$ . If  $I_0$  is empty, then we use the second part of Lemma 5.1 to obtain  $\chi_i \in -\Sigma_M(Q)$  for all  $i \geq 2$  when  $\chi_i$  is defined.

Note that  $r_i(\partial(e^3))$  is the boundary of the union  $U$  of 3-cells consisting of  $e^3$  and some prisms  $e_{a,b_i}^3$  glued to the squares  $e_{a,a_i}^2$  (or  $e_{a_i,a}^2$  if  $a_i < a$ ) of  $\partial(e^3)$  with support  $\{q_1, q_i\}$ ,  $i \in I_0$ . Consider a new 3-dimensional figure  $U_0$  obtained from  $U$  by gluing a prism  $e_{a,b_i}^3$  to the square  $e_{a,a_i}^2$  (or  $e_{a_i,a}^2$ ) with support  $\{q_1, q_i\}$  where  $i \in \{2, \dots, s\} \setminus I_0$  and  $q_i$  is on the boundary of  $v_0 + B(\rho, z_0)$ . All elements from the support of  $e_b^2$  except  $q_i$  are in the open ball with centre  $y_i$  and radius  $z_0$  (this is possible because  $[\chi_i] \in -\Sigma_M(Q)$ ). Note that the global diameter of  $e_{a,b_i}^3$  is smaller than  $\rho$  and so  $\partial(U)$  is contractible in  $X_{\rho'}$  if and only if  $\partial(U_0)$  is contractible in  $X_{\rho'}$ .

By the construction of the additional 3-dimensional pushing cells in  $X$  given in Section 4, the top (bottom) of  $\partial(U_0)$  is homotopic via a 3-dimensional pushing cell to a union  $C_0$  of 2-cells with the property that the support of every cell in  $C_0$  is in the interior of  $v_0 + B(\rho, z_0)$ , that is, no element from the support is on the boundary  $v_0 + B(\rho, z_0)$  (see Figure 7 and Figure 5). Then the boundary of  $U_0$  is the boundary

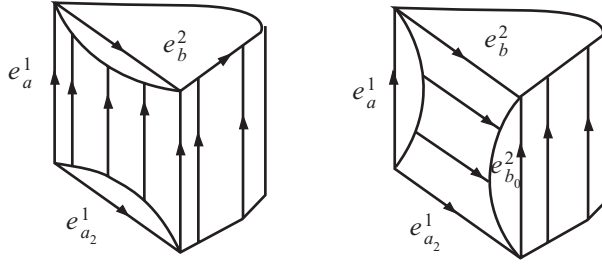


FIGURE 8. Image of the boundary of a special prism under  $r_i$ .

of the union of the two pushing 3-cells to the top and bottom of  $U_0$  together with a union  $U_1$  of 3-cells where every 3-cell in  $U_1$  is the product of  $e_a^1$  with a cell from  $C_0$ . Because pushing 3-cells are contained in  $X_{\rho'}$ , the boundary of  $U_0$  is contractible in  $X_{\rho'}$  if and only if this is true for the boundary of  $U_1$ . Now note that the boundary of  $U_1$  is the boundary of the 3-cells in the product of  $C_0$  and  $e_a^1$ , and that these 3-cells have support contained in

$$\{\text{the interior of } v_0 + B(\rho, z_0)\} \cup \{\text{the point } q_1 \text{ from the boundary of } v_0 + B(\rho, z_0)\},$$

and are hence contained in  $X_{\rho'}$ .

*Case 1.2:* We assume that  $e^3$  is a special prism, that is, the global diameter of only one square of  $\partial(e^3)$  is  $\rho$ . Without loss of generality, we assume that the global diameter of  $\{q_1, q_2\}$  is  $\rho$ , and let  $e_{a_2}^1$  be the unique edge from  $\partial(e_b^2)$  with support  $q_2$ . Then  $r_i(\partial(e^3))$  is of one of the types shown in Figure 8.

In the first case (on the left-hand side of Figure 8), we use the method for Case 1.1. As in Case 1.1, we glue additional prisms  $e_{a,b_i}^3$  to the squares  $e_{a,a_i}^2$ , where  $e_{a_i}^1$  is an edge of the boundary of  $e_b^2$  with support  $q_i$  on the boundary of  $v_0 + B(\rho, z_0)$  for some  $i \geq 3$ . Thus we obtain a new figure  $U_0$ : the union of  $r_i(\partial(e^3))$  with the new prisms. Note that as  $(q_2 - y_2)/(|q_2 - y_2|) \in -\Sigma_M(Q)$  for sufficiently large  $z_0$  for all  $q_i$  from  $\text{supp}(e_b^2) \cap \partial(v_0 + B(\rho, z_0))$ , we have  $(q_i - y_i)/(|q_i - y_i|) \in -\Sigma_M(Q)$ . Then the top and bottom of  $U_0$  are homotopic via a 3-dimensional pushing cell to a union  $C_0$  of 2-dimensional cells with support strictly inside  $v_0 + B(\rho, z_0)$ . Then we continue as in Case 1.1.

Now we consider the second case. Note that  $r_i(\partial(e^3))$  is the boundary of the union of the 3-cell  $e^3$  and the prism that is the product of  $e_{b_0}^2$  and  $e_{a_2}^1$ . We glue prisms to all other squares of the boundary of  $e^3$  to obtain a union  $U_0$  of 3-cells. The prisms we glue are products of the pushing cell  $e_{b_0}^2$  with the edges from  $\partial(e_b^2) \setminus e_{a_2}^1$  and so have global diameter smaller than  $\rho$ ; here  $e_{b_0}^2$  is the pushing cell used in the definition of  $r_i(e_{a,a_2}^2)$  to push  $e_a^1$ . Thus  $r_i(\partial(e^3))$  is contractible in  $X_{\rho'}$  if and only if the boundary of  $U_0$  is contractible in  $X_{\rho'}$ . Now note that the boundary of  $U_0$  is also the boundary of a union  $U_1$  of prisms of global diameter smaller than  $\rho$ . These prisms are products of  $e_b^2$  with edges of the boundary of  $e_{b_0}^2$  different from  $e_a^1$  (see Figure 9). In particular, the boundary of  $U_0$  is contractible in  $X_{\rho'}$ . Thus we can extend  $r_i$  to all special prisms in  $X_{\rho}$ .

*Case 1.3:* Suppose that  $u_1 = (q_1 - y_1)/(|q_1 - y_1|) \in -\Sigma_M(Q)$ . Then there is a  $b_0 \in B$  such that  $e_{b_0}^2$  is a pushing 2-cell that gives a homotopy between the edge  $e_a^1$  and a

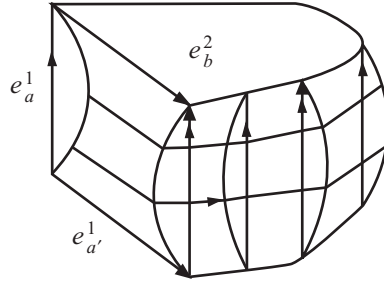


FIGURE 9. Surrounding a special prism with prisms.

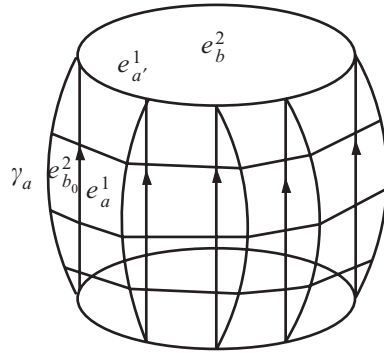


FIGURE 10. Surrounding a prism with special prisms.

path  $\gamma_a$ ; the support of  $\gamma_a$  is in the open ball with centre  $y_1$  and radius  $z_0 - v/2$  (see Figure 10). We attach to every square  $e_{a,a'}^2$  in the boundary of the prism  $e^3$  a special prism, a product of the corresponding  $e_{b_0}^2$  and  $e_{a'}^1$ , and obtain the union  $U_0$  of all these 3-cells. Now note that the boundary of  $U_0$  is also the boundary of a union  $U_1$  of prisms of global diameter smaller than  $\rho$ . The prisms in  $U_1$  are products of  $e_b^2$  with the edges of the path  $\gamma_a$ . This shows that the boundary of  $e^3$  is the boundary of a union of special prisms from  $X_\rho$  and prisms already contained in  $X_{\rho'}$ . Since  $r_i$  is defined on these 3-cells, we can extend  $r_i$  to  $e^3$ .

*Case 2:* We assume that  $e^3 = e_{a_1, a_2, a_3}^3$  is a 3-cube of global diameter  $\rho$  of support  $\{q_1, q_2, q_3\}$ . Furthermore, we assume that  $\{q_1, q_2, q_3\} \subseteq v_0 + B(\rho, z_0)$ . For every  $q_i$  on the boundary of  $B(\rho, z_0)$ , we define  $y_i$  to be the projection of  $q_i$  to  $v_0 + I(\rho, z_0)$ . Then, by the first part of Lemma 5.1 for  $m = 3$ , there exists for some  $i_0 \in \{1, 2, 3\}$

$$\frac{q_{i_0} - y_{i_0}}{|q_{i_0} - y_{i_0}|} \in -\Sigma_M(Q).$$

Thus there is a pushing 2-cell  $e_b^2$  that gives a homotopy between the edge

$$e_{a_{i_0}}^1$$

and a path  $\gamma$  whose support is in the open ball with centre  $y_{i_0}$  and radius  $z_0 - v/2$ . Now we glue prisms  $e_{a,b}^3$  to every square of the boundary of  $e^3$  with edges with labels  $a$  and  $a_{i_0}$  to obtain a union  $U_0$  of 3-cells (see Figure 11). Note that the prisms that we have glued on are special prisms of global diameter at most  $\rho$ , and so  $r_i$  is already defined on them by Case 1.2, previously considered. Note that the boundary

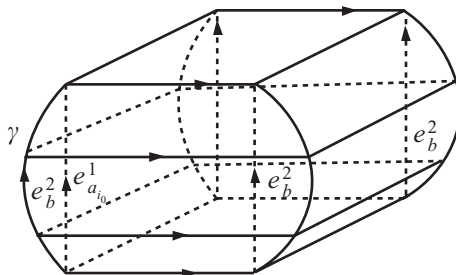


FIGURE 11. Surrounding a cube with special prisms.

of  $U_0$  is also the boundary of a union  $U_1$  of cubes of smaller global diameter. These cubes are products of the square of  $\partial(e^3)$  not containing

$$e_{a_{i_0}}^1$$

with the edges from  $\gamma$ . This shows that  $\partial(e^3)$  is the boundary of the union of 3-cells of  $X_\rho$  on which  $r_i$  is already defined. Thus we can extend  $r_i$  to  $e^3$ .

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