On the Homotopy Type of CW-Complexes with Aspherical Fundamental Group

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Abstract

This paper is concerned with the homotopy type distinction of finite CW-complexes. A (G, n)-complex is a finite n-dimensional CW-complex with fundamental-group G and vanishing higher homotopy-groups up to dimension $n - 1$. In case G is an n-dimensional group there is a unique (up to homotopy) (G, n)-complex on the minimal Euler-characteristic level $\chi_{min}(G, n)$. For every n we give examples of n-dimensional groups G for which there exist homotopically distinct (G, n)-complexes on the level $\chi_{min}(G, n) + 1$.

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1 Introduction

This paper is concerned with the homotopy type distinction of CW-complexes. A CW-complex is called *aspherical* if all its higher homotopy groups vanish. A (G, n)-complex is a finite n-dimensional CW-complex with fundamental-group G and vanishing higher homotopy-groups up to dimension $n - 1$. Note that a (G, n)-complex is the n-skeleton of an aspherical complex that has finite n-skeleton. Note also that a $(G, 2)$-complex is simply a finite 2-complex with fundamental group G. For a given group G we are investigating the question

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whether there can be homotopically distinct (G,n)-complexes with the same Euler-characteristic.

Suppose that X is a finite n-dimensional CW-complex and denote by c_k the number of k-cells of X. By the *directed Euler-characteristic of* X, $\chi_d(X)$, we mean the alternating sum $\sum_{i=0}^{n}(-1)^{n-i}c_i$. If X is a (G,n)-complex then it is not difficult to see that $\chi_d(X)$ is bounded from below by $\sum_{i=0}^{n}(-1)^{n-i}\mathrm{dim}H_i(G,\mathbb{Q})$, a constant that only depends on the homology of G. Thus we can define $\chi_{\min}(G,n)$ to be the minimal directed Euler-characteristic that can occur.

We say a group G is *n-dimensional* if it is the fundamental-group of a finite n-dimensional aspherical complex and there is no such complex of smaller dimension. Every (G,n)-complex of minimal directed Euler-characteristic $\chi(G,n)$ of a n-dimensional group G is aspherical. Hence up to homotopy there is a unique (G,n)-complex of minimal directed Euler-characteristic (Theorem 3). We show in this paper that if G is a n-dimensional group that contains the trefoil-group as a retract, then there are homotopically inequivalent (G,n)-complexes with directed Euler-characteristic $\chi_{min}(G,n)+1$ (Theorem 7). For the trefoil-group itself this was observed by Dunwoody [3]. See also the interesting generalizations obtained by Lustig [10]. We also outline a program for constructing different homotopy types of 2-complexes on Euler-characteristic levels higher that $\chi_{min}(G,n)+1$ (Theorem 8 and Section 5). Additional information on the classification of homotopy types and related topics can be found in the excellent book [7].

2 Presentations of stably-free modules

Let R be a unitary ring. A R-module P is called stably-free if there are natural numbers m and n so that $P\oplus R^m$ is isomorphic to R^n. Another way to say this is that a stably-free module is the kernel of an epimorphism $\phi : R^n \to R^m$. By a *splitting of* ϕ we mean a homomorphism $s : R^m \to R^n$ such that $\phi \circ s$ is the identity.

Lemma 1 *Let P be the kernel of an epimorphism $\phi : R^n \to R^m$. Choose a basis $e_1, \ldots, e_n$ of R^n and a splitting s of ϕ. Then P is generated by the elements $e_i - s\circ\phi(e_i)$, $i = 1, \ldots, n$. Furthermore the inclusion induces an isomorphism $P \to R^n/s(R^m)$.*

PROOF. Since every element v of R^n can be uniquely written as

$$v = (v - s\circ\phi(v)) \oplus s\circ\phi(v)$$

we see that $R^n = P \oplus s(R^n)$. Since the elements e_i, $i = 1, \dots, n$ generate R^n and

$$e_i = (e_i - s \circ \phi(e_i)) \oplus s \circ \phi(e_i)$$

we see that the elements $e_i - s \circ \phi(e_i)$, $i = 1, \dots, n$, generate P and that the inclusion induces an isomorphism $P \to R^n/s(R^m)$.

Notice that if $m = 1$ and $\phi(e_i) = \alpha_i \in R$, $i = 1, \dots, n$, every choice of elements β_i, $i = 1, \dots, n$, such that $\sum_{i=1}^n \beta_i \alpha_i = 1$ determines a splitting of ϕ. Indeed, simply define $s(1) = \sum_{i=1}^n \beta_i e_i$.

In the remainder of this section we will discuss Dunwoody's exotic presentation for the trefoil group T (see [3]). First, T has the well known 1-relator presentation $\langle a, b, \mid a^2 = b^3 \rangle$. Let X be the 2-complex associated with it. Let $r = a^2 b^{-3}$ and denote by N the normal closure of r in the free group on a, b. Dunwoody considers the presentation $\langle a, b \mid u_1, u_2 \rangle$ where $u_1 = rara^{-1}a^2ra^{-2}$ and $u_2 = rbrb^{-1}b^2rb^{-2}b^3rb^{-3}$ and shows that the second homotopy module $\pi_2(X_1)$ of the associated 2-complex X_1 can not be generated by a single element and hence is stably-free but not free. Since the presentation $\langle a, b, \mid a^2 = b^3, 1 \rangle$ gives rise to a 2-complex X_2 with second homotopy module free of rank one, we see that there are homotopically distinct $(T, 2)$-complexes with Euler-characteristic $\chi_{\min}(T, 2) + 1 = 1$.

Using the above Lemma 1 it is not difficult to exhibit generators and a presentation for the module $\pi_2(X_1)$. Let $\alpha_1 = 1 + a + a^2$ and $\alpha_2 = 1 + b + b^2 + b^3$. Consider the cellular chain complex $(C_*(\tilde{X}_1), \partial)$ of the universal covering of X_1. It gives rise to an exact sequence (see [9], Section 3 of Chapter II)

$$0 \to \pi_2(X_1) \to C_2(\tilde{X}_1) \xrightarrow{\phi} \bar{N} \to 0$$

where $\bar{N}$ is the relation-module for the generators a, b of T. It is free of rank 1 and is generated by $r[N, N]$. The second chain group $C_2(\tilde{X})$ has a basis e_1, e_2 consisting of 2-cells that present lifts of the 2-cells in X corresponding to the two relations $u_1 = rara^{-1}a^2ra^{-2}$ and $u_2 = rbrb^{-1}b^2rb^{-2}b^3rb^{-3}$. Furthermore $\phi(e_i) = u_i[N, N] = \alpha_i r[N, N]$, $i = 1, 2$. Lemma 1 and the remark thereafter tell us that every choice of elements $\beta_1, \beta_2 \in \mathbb{Z}T$ such that $\beta_1\alpha_1 + \beta_2\alpha_2 = 1$ gives rise to a splitting of ϕ and hence to explicit generators $e_i - \alpha_i(\beta_1 e_1 + \beta_2 e_2)$ and a presentation for $\pi_2(X_1) = \mathbb{Z}T^2/\beta_1 e_1 + \beta_2 e_2$.

In the following we will compute a particular choice for β_1 and β_2. Note first that $(a - 1)\alpha_1 = a^3 - 1$ and $(b - 1)\alpha_2 = b^4 - 1$. Set $x = a^3$ and $y = b^4$. The elements x and y generate the group. Indeed, $x^3 y^{-3} = a$ and $x^2 y^{-2} = b$. Hence $x - 1$ and $y - 1$ generate the augmentation ideal IT. Since $\alpha_1 - \alpha_2$ augments to -1 we see that $x - 1$, $y - 1$ and $\alpha_1 - \alpha_2$ generate $\mathbb{Z}T$ and hence α_1 and α_2 generate $\mathbb{Z}T$. Now $\alpha_1 - \alpha_2 + 1$ is in the augmentation ideal IT and so we can write it as a linear combination $\alpha_1 - \alpha_2 + 1 = \gamma_1(x-1) + \gamma_2(y-1)$ for certain $\gamma_i \in \mathbb{Z}T$. Solving for 1 we obtain $1 = (\gamma_1(a-1)-1)\alpha_1 + (\gamma_2(b-1)+1)\alpha_2$. So we get a choice for the desired β_i by computing the γ_i and that can be quickly accomplished using the Fox-calculus (see [9], Section 3 of Chapter II).

$$\begin{aligned}
\alpha_1 - \alpha_2 + 1 &= (a-1) - (b+2)(b-1) \\
&= (x^3y^{-3} - 1) - (b+2)(x^2y^{-2} - 1) \\
&= \frac{\partial x^3y^{-3}}{\partial x}(x-1) + \frac{\partial x^3y^{-3}}{\partial y}(y-1) \\
&\quad -(b+2)\left(\frac{\partial x^2y^{-2}}{\partial x}(x-1) + \frac{\partial x^2y^{-2}}{\partial y}(y-1)\right) \\
&= \left(\frac{\partial x^3y^{-3}}{\partial x} - (b+2)\frac{\partial x^2y^{-2}}{\partial x}\right)(x-1) \\
&\quad + \left(\frac{\partial x^3y^{-3}}{\partial y} - (b+2)\frac{\partial x^2y^{-2}}{\partial y}\right)(y-1).
\end{aligned}$$

Let us make the Fox-derivatives explicit, remembering that $a^2 = b^3$ in $\mathbb{Z}T$:

$$\frac{\partial x^3y^{-3}}{\partial x} = 1 + x + x^2 = 1 + a^3 + a^6,$$

$$\frac{\partial x^3y^{-3}}{\partial y} = -x^3y^{-1} - x^3y^{-2} - x^3y^{-3} = -(a + ab^4 + ab^8).$$

Similarly we get

$$\frac{\partial x^2y^{-2}}{\partial x} = 1 + a^3,$$
$$\frac{\partial x^2y^{-2}}{\partial y} = -(b + b^5).$$

Thus we have

$$\gamma_1 = (1 + a^3 + a^6) - (b+2)(1+a^3),$$
$$\gamma_2 = -(a + ab^4 + ab^8) + (b+2)(b+b^5),$$

and hence

$$\beta_1 = ((1 + a^3 + a^6) - (b+2)(1+a^3))(a-1) - 1,$$

$$\beta_2 = (-(a + ab^4 + ab^8) + (b+2)(b+b^5))(b-1) + 1.$$

We summarize our findings in the following

Theorem 2 *Let X_1 be the 2-complex associated with the presentation*

$$\langle a, b \mid u_1, u_2 \rangle$$

for the trefoil group T, where

$$u_1 = rara^{-1}a^2ra^{-2}, u_2 = rbrb^{-1}b^2rb^{-2}b^3rb^{-3},$$

$r = a^2b^{-3}$. *Then the second homotopy-module* $\pi_2(X_1)$ *can not be generated by a single element and hence is stably-free but not free (Dunwoody [3]). It is generated as a submodule of* $C_2(\tilde{X}_1)$ *by* $e_i - \alpha_i(\beta_1 e_1 + \beta_2 e_2)$, $i = 1, 2$. *Furthermore the inclusion* $\pi_2(X_1) \hookrightarrow C_2(\tilde{X}_1)$ *induces an isomorphism* $\pi_2(X_1) = \mathbb{Z}T^2/\beta_1 e_1 + \beta_2 e_2$. *Here*

$$\alpha_1 = 1 + a + a^2, \ \ \alpha_2 = 1 + b + b^2 + b^3$$

$$\beta_1 = ((1 + a^3 + a^6) - (b + 2)(1 + a^3))(a - 1) - 1,$$
$$\beta_2 = (-(a + ab^4 + ab^8) + (b + 2)(b + b^5))(b - 1) + 1.$$

We end this section with a question. Let X be the 2-complex modelled on the standard one-relator presentation of T and X_1 be as in Theorem 2. Let $X_2 = X \vee S^2$. Is $Y_1 = X_1 \vee X_1$ homotopically equivalent to $Y_2 = X_2 \vee X_2$? Note that if $G = T * T$ then $\chi(Y_1) = \chi(Y_2) = \chi_{min}(G, 2) + 2$. So far no pair of homotopically distinct 2-complexes with the same fundamental group and Euler characteristic more than one above the minimal level is known! The question comes down to proving that $\pi_2(Y_1) = \mathbb{Z}G^4/(\beta_1 e_1 + \beta_2 e_2, \beta_1 e_3 + \beta_2 e_4)$ is not free of rank two where β_1 and β_2 are as in Theorem 2.

3 General results and examples

Theorem 3 *Let* G *be a* k*-dimensional group. Up to homotopy there exists a unique* (G, k)*-complex with directed Euler characteristic equal to* $\chi_{\min}(G, k)$.

PROOF. Since we assumed G to be k-dimensional there is a finite aspherical k-dimensional complex X with fundamental group G. Since the homology of X is the homology of the group G we have $\chi_d(X) = \chi_{\min}(G, k)$. Suppose Y is a (G, k)-complex with the same Euler characteristic. We will show that Y is aspherical and hence homotopic to X.

Consider the cellular chain complexes $C_*(\tilde{X})$ and $C_*(\tilde{Y})$ of the universal coverings. It follows from Schanuel's Lemma (see [2]) that $H_k(\tilde{Y}) \oplus A = B$ where

$$A = C_k(\tilde{X}) \oplus C_{k-1}(\tilde{Y}) \oplus C_{k-2}(\tilde{X}) \oplus ...$$

and

$$B = C_k(\tilde{Y}) \oplus C_{k-1}(\tilde{X}) \oplus C_{k-2}(\tilde{Y}) \oplus ...$$

The fact that $\chi_d(X) = \chi_d(Y)$ implies that the free $\mathbb{Z}G$-modules A and B have equal rank, so $H_k(\tilde{Y}) \oplus \mathbb{Z}G^l = \mathbb{Z}G^l$ for some $l \geq 0$. Kaplansky's Theorem (see [7], page 328) now implies that $H_k(\tilde{Y}) = 0$. So Y is indeed aspherical.

Theorem 4 *Let G be a k-dimensional group, $k \geq 3$, and assume P is a stably-free non-free projective module over the group ring $\mathbb{Z}G$ which is the kernel of an epimorphism $\phi : \mathbb{Z}G^n \to \mathbb{Z}G^m$. Then there are (G,k)-complexes X_1 and X_2 with directed Euler-characteristic $\chi_{\min}(G,k)+n-m$ such that $\pi_k(X_1)$ is isomorphic to P and $\pi_k(X_2)$ is free of rank $n-m$. In particular X_1 and X_2 are not homotopically equivalent.*

PROOF. Let X be a finite aspherical complex of dimension k with fundamental group G. Consider the left end of the cellular chain complex $(C_*(\tilde{X}),\partial)$ of the universal covering

$$0 \to C_k(\tilde{X}) \xrightarrow{\partial_k} C_{k-1}(\tilde{X}) \to ...$$

Let $\bar{e}_1,\dots,\bar{e}_l$ be the k-cells in X and denote by e_i a fixed lift of $\bar{e}_i$ in $\tilde{X}$. Then the elements $e_1,\dots,e_l$ form a basis for the $\mathbb{Z}G$-module $C_k(\tilde{X})$ and the kernel of ∂_{k-1} is generated by $\partial_k(e_1),\dots,\partial_k(e_l)$. Remove the k-cells from $\tilde{X}$ and attach $n-m+l$ free G-orbits of k-cells $Gf_1,\dots,Gf_{n-m+l}$ in the following way: suppose that (α_{ij}), $1 \leq i \leq m$, $1 \leq j \leq n$ is a matrix associated with the epimorphism ϕ. Attach gf_s to $g\sum_{j=1}^m \alpha_{js}\partial_k(e_j)$ for $1 \leq s \leq n$ and attach gf_{n+t} to $g\partial_k(e_{m+t})$ for $1 \leq t \leq l-m$ (we assumed $l \geq m$; if not wedge on an appropriate number of k-balls to X). This yields a new complex $\tilde{X}_1$. Note that the new boundary map

$$\partial'_k = \phi \oplus \partial_k : C_k(\tilde{X}_1) = \mathbb{Z}G^n \oplus \mathbb{Z}G^{l-m} \to \partial_k(C_k(\tilde{X})) = \mathbb{Z}G^m \oplus \mathbb{Z}G^{l-m} \subseteq C_{k-1}(\tilde{X}_1)$$

maps the first factor $\mathbb{Z}G^n$ on the left to the first factor $\mathbb{Z}G^m$ on the right via ϕ, and the second factor $\mathbb{Z}G^{l-m}$ on the left to the second factor $\mathbb{Z}G^{l-m}$ on the right via ∂_k. Hence $H_k(\tilde{X}_1) = \ker(\phi) = P$. Let X_1 be the orbit complex obtained from $\tilde{X}_1$ by factoring out the action of G. We have $\pi_k(X_1) = H_k(\tilde{X}_1) = P$. We build a second complex X_2 by wedging $n-m$ k-spheres to X. Note that $\chi_d(X_1) = \chi_d(X_2) = n-m+\chi_d(X)$ and $\pi_k(X_2)$ is a free $\mathbb{Z}G$-module of rank $n-m$. Hence X_1 and X_2 are not homotopy-equivalent.

Let us discuss the case $k=2$. The construction of the complex $\tilde{X}_1$ works just as well but one should notice that because we are restructuring the 2-skeleton this can have an effect on the fundamental group. In fact, it is possible that the complex $\tilde{X}_1$ is not simply-connected and the fundamental group of the quotient complex X_1 might be different from G. However we do have two 2-dimensional chain complexes $C_*(\tilde{X}_1)$ and $C_*(\tilde{X}_2)$ that have the same directed Euler characteristic but are not chain homotopically equivalent because $H_2(\tilde{X}_2)$ is free and $H_2(\tilde{X}_1) = P$, which is not free.

By an *algebraic* (G,n)*-complex* we mean an exact sequence

$$\mathcal{C} : F_n \to ... \to F_0 \to \mathbb{Z} \to 0$$

where the F_i, $i = 1, \ldots, n$ are finitely generated free $\mathbb{Z}G$-modules. If c_i is the rank of the module F_i then the *directed Euler characteristic of* $\mathcal{C}$, $\chi_d(\mathcal{C})$, is the alternating sum $\sum_{i=0}^{n}(-1)^{n-i}c_i$. Of course, if X is a (G,n)-complex then the cellular chain complex $C_*(\tilde{X})$ of the universal covering $\tilde{X}$ is an algebraic (G,n)-complex.

The above discussion yields the following

Theorem 5 *Let G be a 2-dimensional group and assume P is a stably-free non-free projective module over the group ring $\mathbb{Z}G$ which is the kernel of an epimorphism $\phi : \mathbb{Z}G^n \to \mathbb{Z}G^m$. Then there are algebraic $(G,2)$-complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ with directed Euler-characteristic $\chi_{min}(G,2)+n-m$ such that $H_2(\mathcal{C}_1)$ is isomorphic to P and $H_2(\mathcal{C}_2)$ is free of rank $n-m$. In particular $\mathcal{C}_1$ and $\mathcal{C}_2$ are not chain-homotopy equivalent.*

We say a group H is a *retract* of a group G if there are maps $H \xrightarrow{j} G \xrightarrow{p} H$ so that the composition $p \circ j$ is the identity. If M is a finitely generated $\mathbb{Z}G$-module, we denote by $d_G(M)$ the rank, that is the minimal number of generators of M.

Lemma 6 *Suppose H is a retract of G and there exists an epimorphism $\phi : \mathbb{Z}H^n \to \mathbb{Z}H^m$ with kernel P and $d_H(P) > n-m$, i.e. P is a stably-free non-free projective module. Then $\mathbb{Z}G \otimes_H P$ is a stably-free non-free projective module over $\mathbb{Z}G$.*

PROOF. Clearly the induced module $\mathbb{Z}G \otimes_H P$ is the kernel of the induced epimorphism $\mathbb{Z}G \otimes_H \phi : \mathbb{Z}G^n \to \mathbb{Z}G^m$. We view P as a $\mathbb{Z}G$-module via the epimorphism $p : G \to H$. Now the homomorphism $\mathbb{Z}G \otimes_H P \to P$ that sends $g \otimes x$ to $p(g)x$ is an epimorphism. Hence

$$d_G(\mathbb{Z}G \otimes_H P) \geq d_G(P) = d_H(P) > n-m.$$

Hence $\mathbb{Z}G \otimes_H P$ is not free.

Theorem 7 *Suppose G is a k-dimensional group, $k \geq 2$, that contains the trefoil group T as a retract. Then there are homotopically distinct (G,k)-complexes with directed Euler characteristic $\chi_{min}(G,k)+1$. In the case where $k=2$ these are algebraic.*

PROOF. Let X_1 be the 2-complex of Theorem 2. Then $\pi_2(X_1)$ is the kernel of the epimorphism $\phi : \mathbb{Z}T^2 \to \mathbb{Z}T$ given by $\phi(e_1) = 1+a+a^2$ and $\phi(e_2) = 1+b+b^2+b^3$. Dunwoody shows in [3] that $d_T(\pi_2(X_1)) = 2$. In particular $\pi_2(X_1)$ is stably-free but not free. By Lemma 6, $\mathbb{Z}G \otimes_T \pi_2(X_1)$ is a stably-free non-free projective over $\mathbb{Z}G$ that is the kernel of an epimorphism $\mathbb{Z}G \otimes_H \phi : \mathbb{Z}G^2 \to \mathbb{Z}G$. The result follows from Theorems 4 and 5.

Examples.

(1) The group $G = T \times \mathbb{Z}^k$, $k \geq 1$ is $(k+2)$-dimensional and contains T as a retract. Thus there are homotopically distinct $(G, k+2)$-complexes with directed Euler characteristic $\chi_{min}(G, k+2) + 1$.

(2) The group $G = T * \mathbb{Z}$ is a 2-dimensional group which contains T as a retract. Thus there are chain homotopically distinct algebraic $(G, 2)$-complexes with Euler characteristic $\chi_{min}(G, 2) + 1$. The distinct complexes can be geometrically realized. Indeed, if X is the 2-complex modelled on the standard one-relator presentation of T and X_1 is the 2-complex from Theorem 2, then $X_1 \vee S^1$ and $X \vee S^2 \vee S^1$ have the same fundamental group and Euler characteristic but non-isomorphic second homotopy modules.

(3) Since the commutator subgroup $[T, T]$ of the trefoil group is free of rank two we see that T is free-by-cyclic. Indeed, it is not difficult to show that $\langle x, y, t \mid txt^{-1} = y, tyt^{-1} = x^{-1}y \rangle$ presents T. Consider the group G presented by $\langle x, y, z_1, \ldots, z_n, t \mid txt^{-1} = y, tyt^{-1} = x^{-1}y, tz_it^{-1} = w_i \rangle$, where $i = 1, \ldots, n$ and $w_1, \ldots, w_n$ is a basis for the free group on the z_i. Since $G/N = T$ where N is the normal closure of the z_i we see that G contains T as a retract. Thus there are chain homotopically distinct algebraic $(G, 2)$-complexes with Euler characteristic $\chi_{min}(G, 2) + 1$.

We end this section with more comments on the 2-dimensional case. Suppose X is a standard 2-complex modelled on a presentation for the group G, $\mathcal{P} = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$. Then the complex X_1 of Theorem 4 can also be modelled on a presentation and we will now make this presentation explicit. First some notation. Let F be the free group on $x_1, \ldots, x_k$, let R be the normal closure of the relations $r_1, \ldots, r_l$ in F and let $p : F \to G$ be an epimorphism with kernel R. For every $g \in G$ choose an element $\bar{g} \in F$ so that $p(\bar{g}) = g$. Furthermore choose a total ordering on the countable set G. If $r \in R$ and $\alpha = \sum_{i=1}^{t} n_i g_i \in \mathbb{Z}G$, where $g_1 < \ldots < g_t$, then we define ${}^{\alpha}r = \bar{g}_1 r^{n_1} \bar{g}_1^{-1} \ldots \bar{g}_t r^{n_t} \bar{g}_t^{-1}$.

Let $\phi : \mathbb{Z}G^n \to \mathbb{Z}G^m$ be an epimorphism and let (α_{ij}) be a matrix for ϕ, $1 \leq i \leq m$, $1 \leq j \leq n$. Define $u_s = {}^{\alpha_{1s}}r_1 \ldots {}^{\alpha_{ms}}r_m$, $s = 1, \ldots, n$. Let $\mathcal{P}_\phi = \langle x_1, \ldots, x_k \mid u_1, \ldots, u_n, r_{m+1}, \ldots, r_{m+(l-m)} \rangle$.

Theorem 8 *Let $\phi : \mathbb{Z}G^n \to \mathbb{Z}G^m$ be an epimorphism, X be an aspherical 2-complex modelled on the presentation $\mathcal{P} = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ for G and X_1 be the 2-complex modelled on the presentation $\mathcal{P}_\phi$. If $\mathcal{P}_\phi$ also presents G then $\pi_2(X_1)$ is isomorphic to the kernel of ϕ. In particular, if the kernel of ϕ is not free of rank $n - m$ then the 2-complexes X_1 and $X_2 = X \vee S^2 \ldots \vee S^2$*

($n-m$ 2-spheres) are not homotopically equivalent.

PROOF. Build a 2-complex $\tilde{X}_1$ from the 1-skeleton of $\tilde{X}$ and the epimorphism ϕ as in the proof of Theorem 4. Note that by construction $H_2(\tilde{X}_1)$ is the kernel of ϕ. Observe that the orbit complex $\tilde{X}_1/G$ is X_1. The assumption that the fundamental group of X_1 is G implies that $\tilde{X}_1$ is the universal covering of X_1. Hence $\pi_2(X) = H_2(\tilde{X}) = ker(\phi)$.

In Dunwoody's example the conditions in the theorem are satisfied: The epimorphism $\phi : \mathbb{Z}T^2 \to \mathbb{Z}T$ is given by the matrix (α_1, α_2), where $\alpha_1 = 1+a+a^2$ and $\alpha_2 = 1+b+b^2+b^3$. The 2-complex X is modelled on the presentation $\mathcal{P} = \langle a, b \mid r \rangle$, $r = a^2b^{-3}$. Since the kernel of ϕ is not free of rank 1 and $\mathcal{P}_\phi = \langle a, b \mid {}^{\alpha_1}r, {}^{\alpha_2}r \rangle$, ${}^{\alpha_1}r = rara^{-1}a^2ra^{-2}$, ${}^{\alpha_2}r = rbrb^{-1}b^2rb^{-2}b^3rb^{-3}$, does present the trefoil group T, the complexes X_1 modelled on $\mathcal{P}_\phi$ and $X_2 = X \vee S^2$ are not homotopically equivalent.

4 An application

If M is a finitely generated $\mathbb{Z}G$-module we denote by $d_G(M)$ the rank of M (that is the minimal number of generators). Let $\mathcal{C}$ be an algebraic $(G, 1)$ complex. So $\mathcal{C}$ is an exact sequence

$$\mathcal{C} : F_1 \to F_0 \to \mathbb{Z} \to 0$$

where the F_i, $i = 0, 1$ are finitely generated free $\mathbb{Z}G$-modules. The module $H_1(\mathcal{C})$ is a generalized relation module for the group G. It has been known for a long time that the difference $d_G(H_1(\mathcal{C})) - \chi_d(\mathcal{C})$ is an invariant for G in case that G is finite. Dunwoody's exotic presentations show that this result does not extend to finitely presented groups. A natural question is whether similar results hold in higher dimensions. Here is the complete answer for finite groups.

Theorem 9 *(Gruenberg [6]) Let G be a finite group and $\mathcal{C}$ be an algebraic (G, n) complex, $n \geq 1$. Then the difference*

$$d_G(H_n(\mathcal{C})) - \chi_d(\mathcal{C})$$

is an invariant of G except when $\mathbb{Z}G$ fails to allow cancellation and $\mathbb{Z}$ has projective period $4k$, $k \geq 1$, and $n \equiv 2(mod\ 4)$.

The exceptional case occurs for example when G is the generalized quaternion group of order 32. Here $\mathbb{Z}$ has projective period 4 over the group ring $\mathbb{Z}G$.

Theorem 10 *Let T be the trefoil group and $G = T \times \mathbb{Z}^{k-2}$, $k-2 \geq 0$. Let $\mathcal{C}$ be an algebraic (G,k)-complex. Then the difference*

$$d_G(H_k(\mathcal{C})) - \chi_d(\mathcal{C})$$

is not independent of the choice of $\mathcal{C}$.

PROOF. Let X_1 be the 2-complex of Theorem 2. Dunwoody shows in [3] that $d_T(\pi_2(X_1)) = 2$. In particular $\pi_2(X_1)$ is stably-free but not free. By Lemma 6, $\mathbb{Z}G \otimes_T \pi_2(X_1)$ is a stably-free non-free projective over $\mathbb{Z}G$. The result follows from Theorem 4 in case $k-2 \geq 1$ and Theorem 5 in case $k-2=0$.

5 Questions and open ends

Some motivation for the present paper came from the following open question: Can there be homotopically distinct 2-complexes X_1 and X_2 with the same fundamental group G and Euler characteristic $\chi(X_1) = \chi(X_2) > \chi(G,2)+1$? Dunwoody's examples (and Lustig's generalizations) all have Euler-characteristic exactly one above the minimal level. We believe that our techniques will eventually lead to a positive answer for the above question. The following line of approach seems promising to us.

Let G be a 2-dimensional aspherical group. Choose left module generators $\alpha_1, \ldots, \alpha_n$ $n \geq 3$, of $\mathbb{Z}G$. This determines an epimorphism $\phi : \mathbb{Z}G^n \to \mathbb{Z}G$, where $\phi(e_i) = \alpha_i$, $i = 1, \ldots, n$. Suppose that the presentation $\mathcal{P}_\phi$ does define the group G. Let X_1 be the 2-complex modelled on $\mathcal{P}_\phi$. We know from Theorem 7 that $\pi_2(X_1)$ is isomorphic to the kernel of ϕ. In order to compute the minimal number of generators for $\pi_2(X_1)$ we choose elements $\beta_i \in \mathbb{Z}G$, $i = 1, \ldots, n$ so that $\sum_{i=1}^n \beta_i\alpha_i = 1$. Then $\pi_2(X_1)$ is isomorphic to $M = \mathbb{Z}G^n/\beta_1 e_1 + \ldots + \beta_n e_n$. One can now try to find a quotient of that module for which rank computations can be carried out. If one finds that $d_G(M) > n-1$, then $\pi_2(X_1)$ is not free and hence X_1 is not homotopically equivalent to $X_2 = X \vee S^2 \vee \ldots \vee S^2$ (with n 2-spheres added to X).

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