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Building resolutions

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Abstract

We consider how a resolution of an abelian group M over $\mathbb{Z}$ could be lifted to a free resolution of the trivial module R over $R[M]$, where R is the field of the rationals. The extended resolution is defined in terms of the exterior and divided powers algebras. Furthermore if the resolution of M is in fact a free resolution over $\mathbb{Z}[G]$ for some group G then the extended resolution will provide a free resolution of the augmentation ideal of $R[M]$ over $R[M \rtimes G]$. Furthermore if R is a subring of the rationals containing $\mathbb{Z}$ and all $j \leqslant i$ are invertible in R then the extended complex can be defined up to dimension $(i+1)$ and is exact up to dimension i.

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1. Introduction

The main purpose of this paper is to show how a resolution of a module M over an integral group algebra $\mathbb{Z}[G]$ could be lifted to a resolution $\mathcal{Q}$ of the trivial module R over $R[M]$ that is endowed with G-action, where R is the ring of the rational numbers $\mathbb{Q}$. In fact all terms of the lifted resolution except the zero term will be free $R[M \rtimes G]$-modules. If we want to construct only the first $(i+1)$-terms of a 'lifted' resolution it is sufficient to require that R is a subring of $\mathbb{Q}$ that contains $\mathbb{Z}$ and every $1 \leqslant j \leqslant i$ is invertible in R. Thus the lifted resolution will be exact up to dimension i.

The idea of lifting a resolution of M was suggested in [3], where the case G finitely generated abelian, $R = \mathbb{Z}$ is considered. There a topological approach is used to show that

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in small dimensions the complex given by Theorem A is exact. This turns out to be helpful in a low dimensional case of the FP_m-Conjecture for metabelian groups [2].

Our interest in lifting resolutions was originally motivated by attempts to extend the results from [3] to higher dimensional cases. Still we believe that the construction of lifted resolutions deserves its own attention, separately from the FP_m-Conjecture. The complex $\mathcal{Q}$ can be viewed as a generalisation of the complex $D(F)$ considered in [9].

The main results of this paper, Corollaries 1 and 2, will be formulated in Section 2 and proved in Section 4. The proof of Theorem 1 is based on spectral sequence arguments. In Section 6 we restrict to the case $G = 1$ and work with coefficients $\mathbb{Z}$. This will give a new approach to a non-functorial description of the homology of abelian groups. More about the homology of abelian groups can be found in [7, Theorem C], [1,4,6].

2. Preliminaries

2.1. Divided powers

In this section we consider a free abelian group V. Throughout the paper if not otherwise stated all tensor and exterior powers are over the ring of integers $\mathbb{Z}$. By definition the ith divided power $\widetilde{S}^i(V)$ of V is $\{\lambda \in \otimes^i V \mid \sigma(\lambda) = \lambda \text{ for all } \sigma \in S_i\}$, note in general it is not isomorphic to the ith symmetric power of V. The divided powers algebra $\Gamma(V) = \bigoplus_{i \geqslant 0} \widetilde{S}^i(V)$ is equipped with symmetric multiplication $*$ that is defined on the whole tensor algebra $T(V) = \bigoplus_{i \geqslant 0}(\otimes^i V)$ by

$$(v_1 \otimes \cdots \otimes v_k) * (v_{k+1} \otimes \cdots \otimes v_{k+s}) = \sum_{(k,s)\text{-shuffle } \sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k+s)}.$$

A (k, s)-shuffle σ is an element of the symmetric group S_{k+s} such that $\sigma(1) < \sigma(2) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+s)$. $\Gamma(V)$ satisfies the axioms of a divided powers algebra with only even degrees: it is a commutative graded $\mathbb{Z}$-algebra where the elements of $\widetilde{S}^j(V)$ have degree sj for some fixed even number s and such that for every x of degree i there is an element $x^{(k)}$ of degree ki and

$$x^{(0)} = 1, \qquad x^{(1)} = x, \qquad x^{(k)} * x^{(j)} = (k, j)x^{(k+j)} \quad \text{where } (k, j) = \frac{(k+j)!}{k!j!};$$

$$(x + y)^{(k)} = \sum_{i+j=k} x^{(i)} * y^{(j)} \quad \text{(Leibniz formula)};$$

$$\bigl(x^{(j)}\bigr)^{(k)} = (j, j-1)(2j, j-1)\cdots\bigl((k-1)j, j-1\bigr)x^{(kj)};$$

$$(x * y)^{(k)} = x^k * y^{(k)} \quad \text{for } y \neq 1.$$

By [1, Expose 8, Proposition 4], $\Gamma(V)$ (denoted $S(V)$ there) is a divided powers algebra with

$$x^{(k)} = \underbrace{x \otimes \cdots \otimes x}_{k \text{ times}} \quad \text{for } x \in V.$$

2.2. Koszul complexes and homology groups

Suppose B is a free abelian group with a linearly ordered basis X. Consider the exact Koszul complex

$$\cdots \to \mathbb{Z}[B] \otimes \wedge^i B \to \mathbb{Z}[B] \otimes \wedge^{i-1} B \to \cdots \to \mathbb{Z}[B] \otimes B \to \mathbb{Z}[B] \to \mathbb{Z} \to 0$$

with differential

$$\partial(x_1 \wedge \cdots \wedge x_i) = \sum_{1 \leqslant j \leqslant i} (-1)^{j-1}(x_j - 1) \otimes \bigl(x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_i\bigr)$$

for $x_1 < \cdots < x_i$ in X. This complex can be viewed as free resolution over $\mathbb{Z}[B]$ of the trivial module $\mathbb{Z}$ and we use it to calculate the homologies $H_i(B, \mathbb{Z})$. The isomorphism

$$H_i(B, \mathbb{Z}) \simeq \wedge^i B = \wedge^i H_1(B, \mathbb{Z})$$

given by the Koszul resolution is natural (see [5, Chapter 5, Theorem 6.4i]) and the product in $H_*(B, \mathbb{Z})$ induced by the exterior product in the Koszul resolution coincides with the Pontryagin product (see [5, Chapter 5.5]).

3. Definition of the lifted complex

Let M be an abelian group, i.e., $\mathbb{Z}$-module. We assume now that

$$\cdots \to \mathbb{Z}\mathcal{A}_i \xrightarrow{d_i} \mathbb{Z}\mathcal{A}_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} \mathbb{Z}\mathcal{A}_1 \xrightarrow{d_1} M \to 0$$

is a resolution of M over $\mathbb{Z}$, where $\mathbb{Z}\mathcal{A}_j$ is the free $\mathbb{Z}$-module with basis $\mathcal{A}_j$. We aim to define a complex

$$\mathcal{Q}\colon \cdots \to Q_m \to Q_{m-1} \to \cdots \to Q_0 = R[M] \to R \to 0$$

over $R[M]$ and a filtration $\{\mathcal{Q}^{(j)}\}_{j \geqslant 1}$ of subcomplexes of $\mathcal{Q}$, i.e.,

$$\mathcal{Q}^{(1)} \subseteq \cdots \subseteq \mathcal{Q}^{(j)} \subseteq \mathcal{Q}^{(j+1)} \subseteq \cdots, \quad \mathcal{Q} = \bigcup_{j \geqslant 1} \mathcal{Q}^{(j)}.$$

First we define the underlying module structure of $\mathcal{Q}^{(j)}$. If j is odd

$$\mathcal{Q}^{(j)}_m = \bigoplus_{\sum_{t\leqslant j} t.i_t = m;\, i_t\geqslant 0} \bigl(R[M]\otimes \wedge^{i_1}(\mathbb{Z}\mathcal{A}_1)\otimes \widetilde{S}^{i_2}(\mathbb{Z}\mathcal{A}_2)\otimes \wedge^{i_3}(\mathbb{Z}\mathcal{A}_3)\otimes\cdots\otimes \wedge^{i_j}(\mathbb{Z}\mathcal{A}_j)\bigr).$$

If $j \geqslant 2$ is even

$$\mathcal{Q}^{(j)}_m = \bigoplus_{\sum_{t\leqslant j} t.i_t = m,\, i_t\geqslant 0} \bigl(R[M]\otimes \wedge^{i_1}(\mathbb{Z}\mathcal{A}_1)\otimes \widetilde{S}^{i_2}(\mathbb{Z}\mathcal{A}_2)\otimes \wedge^{i_3}(\mathbb{Z}\mathcal{A}_3)\otimes\cdots\otimes \widetilde{S}^{i_j}(\mathbb{Z}\mathcal{A}_j)\bigr).$$

Note that in the definition of $\mathcal{Q}^{(j)}_m$ the exterior and divided powers alternate and the elements of $\widetilde{S}^{i_{2k}}(\mathbb{Z}\mathcal{A}_{2k})$ and $\wedge^{i_{2k-1}}(\mathbb{Z}\mathcal{A}_{2k-1})$ have degrees $i_{2k}2k$ and $i_{2k-1}(2k-1)$, respectively. Furthermore $\mathcal{Q}$ is equipped with a strictly anticommutative product whose restriction on the exterior powers is the wedge product and the restriction on the divided powers is the symmetric product $*$. More precisely for j odd and

$$\begin{aligned} &f\otimes\lambda_1\otimes\lambda_2\otimes\cdots\otimes\lambda_j \\ &\quad\in R[M]\otimes \wedge^{i_1}(\mathbb{Z}\mathcal{A}_1)\otimes \widetilde{S}^{i_2}(\mathbb{Z}\mathcal{A}_2)\otimes \wedge^{i_3}(\mathbb{Z}\mathcal{A}_3)\otimes\cdots\otimes \wedge^{i_j}(\mathbb{Z}\mathcal{A}_j), \\ &g\otimes\mu_1\otimes\mu_2\otimes\cdots\otimes\mu_j \\ &\quad\in R[M]\otimes \wedge^{k_1}(\mathbb{Z}\mathcal{A}_1)\otimes \widetilde{S}^{k_2}(\mathbb{Z}\mathcal{A}_2)\otimes \wedge^{k_3}(\mathbb{Z}\mathcal{A}_3)\otimes\cdots\otimes \wedge^{k_j}(\mathbb{Z}\mathcal{A}_j), \end{aligned}$$

we define

$$\begin{aligned} &(f\otimes\lambda_1\otimes\lambda_2\otimes\cdots\otimes\lambda_j)(g\otimes\mu_1\otimes\mu_2\otimes\cdots\otimes\mu_j) \\ &\quad= (fg)(-1)^{\varepsilon}\otimes(\lambda_1\wedge\mu_1)\otimes(\lambda_2*\mu_2)\otimes\cdots\otimes(\lambda_j\wedge\mu_j), \end{aligned}$$

where $\varepsilon = \sum_{1\leqslant t<r\leqslant (j+1)/2} k_{2t-1} i_{2r-1}$. As the elements of the divided powers are central for $\lambda_{j+1}\in \widetilde{S}^{i_{j+1}}(\mathbb{Z}\mathcal{A}_{j+1})$, $\mu_{j+1}\in \widetilde{S}^{k_{j+1}}(\mathbb{Z}\mathcal{A}_{j+1})$ we have

$$\begin{aligned} &(f\otimes\lambda_1\otimes\lambda_2\otimes\cdots\otimes\lambda_{j+1})(g\otimes\mu_1\otimes\mu_2\otimes\cdots\otimes\mu_{j+1}) \\ &\quad= \bigl((f\otimes\lambda_1\otimes\lambda_2\otimes\cdots\otimes\lambda_j)(g\otimes\mu_1\otimes\mu_2\otimes\cdots\otimes\mu_j)\bigr)\otimes(\lambda_{j+1}*\mu_{j+1}) \\ &\quad= (fg)(-1)^{\varepsilon}\otimes(\lambda_1\wedge\mu_1)\otimes(\lambda_2*\mu_2)\otimes\cdots\otimes(\lambda_j\wedge\mu_j)\otimes(\lambda_{j+1}*\mu_{j+1}). \end{aligned}$$

The multiplicative structure of $\mathcal{Q}^{(j)}$ induces a multiplicative structure on $\mathcal{Q}$.

We want to construct the differential of the complex $\mathcal{Q} = \bigcup_j \mathcal{Q}^{(j)}$ inductively on j such that for $\lambda_1\in Q_{m_1}$, $\lambda_2\in Q_{m_2}$

$$\partial(\lambda_1.\lambda_2) = \partial(\lambda_1).\lambda_2 + (-1)^{\deg(\lambda_1)}\lambda_1.\partial(\lambda_2), \tag{1}$$

where $\deg(\lambda_1) = m_1$ is the degree of λ_1. A complex satisfying (1) is called a *DG* (differential graded) ring. First we define the differential of $\mathcal{Q}^{(1)}$ by

$$\partial(a_1 \wedge \cdots \wedge a_i) = \sum_{1 \leqslant k \leqslant i} (-1)^{k-1}\bigl(d_1(a_k) - 1\bigr) \otimes a_1 \wedge \cdots \wedge \hat{a}_k \wedge \cdots \wedge a_i$$

for $a_1 < \cdots < a_i \in \mathcal{A}_1$ for some fixed linear order $<$ in $\mathcal{A}_1$. Note that by construction for $j \geqslant 1$

$$\mathcal{Q}_m^{(j+1)} = \begin{cases} \bigoplus_{m_1+(j+1).m_2=m} \mathcal{Q}_{m_1}^{(j)} \otimes \wedge^{m_2}(\mathbb{Z}\mathcal{A}_{j+1}) & \text{if } j+1 \text{ is odd,} \\ \bigoplus_{m_1+(j+1).m_2=m} \mathcal{Q}_{m_1}^{(j)} \otimes \widetilde{S}^{m_2}(\mathbb{Z}\mathcal{A}_{j+1}) & \text{if } j+1 \text{ is even.} \end{cases}$$

In addition to (1) we want the differential of $\mathcal{Q}$ to have the following properties

$$\partial(\mathbb{Z}\mathcal{A}_{j+1}) \subseteq \mathcal{Q}_j^{(j)} \cap \operatorname{Ker}\partial, \quad \partial|_{\mathbb{Z}\mathcal{A}_{j+1}} \equiv d_{j+1} \quad \text{modulo } \partial\bigl(\mathcal{Q}^{(j)}\bigr). \tag{2}$$

We note that the way we construct the differential by induction on j it is not unique but the induced differential on $\mathcal{Q}^{(j+1)}/\mathcal{Q}^{(j)}$ is unique. Secondly the existence of a differential with property (2) should be justified. To do so we assume we have constructed the differential with the above properties on all $\mathcal{Q}_t^{(j)}$ for $t \leqslant m-1$, where $(m-1)!$ is invertible in R. Furthermore we assume that j is sufficiently small, $j \leqslant m-1$. The following theorem calculates the homologies of $\mathcal{Q}^{(j)}$ up to dimension $m-1$. As a corollary of Theorem 1 we obtain

$$H_j\bigl(\mathcal{Q}^{(j)}\bigr) \simeq \operatorname{Ker} d_j \otimes R \simeq \operatorname{Im} d_{j+1} \otimes R.$$

Then using (2) to extend the differential from $\mathcal{Q}^{(j)}$ to $\mathcal{Q}^{(j+1)}$ we need to define only the differential on $\mathbb{Z}\mathcal{A}_{j+1}$. We do it in such a way that the composition

$$\mathbb{Z}\mathcal{A}_{j+1} \xrightarrow{\partial} \mathcal{Q}_j^{(j)} \cap \operatorname{Ker}\partial \to H_j\bigl(\mathcal{Q}^{(j)}\bigr)$$

is the map

$$d_{j+1} : \mathbb{Z}\mathcal{A}_{j+1} \to \operatorname{Im} d_{j+1} \subseteq \operatorname{Im} d_{j+1} \otimes R \simeq H_j\bigl(\mathcal{Q}^{(j)}\bigr).$$

Theorem 1. *Let j, m be fixed positive integers for which the differential of $\mathcal{Q}^{(j)}$ is defined, satisfies* (1) *and* (2) *and $j \leqslant m-1$. Suppose further that R is a subring of $\mathbb{Q}$ containing the integers $\mathbb{Z}$ such that every element $1 \leqslant i \leqslant m-1$ is invertible in R. Then for $0 \leqslant t \leqslant m-1$ we have*

(1) *If j is odd,*

$$H_t\bigl(\mathcal{Q}^{(j)}\bigr) = \begin{cases} \wedge^{\frac{t}{j}}(\operatorname{Ker} d_j) \otimes R, & \frac{t}{j} \in \mathbb{Z},\ 0 < t \leqslant m-1, \\ 0, & t = 0 \text{ or } \frac{t}{j} \notin \mathbb{Z},\ t \leqslant m-1. \end{cases}$$

(2) *If* $j \geqslant 2$ *is even,*

$$H_t\bigl(\mathcal{Q}^{(j)}\bigr) = \begin{cases} \widetilde{S}^{\frac{t}{j}}(\operatorname{Ker} d_j) \otimes R, & \frac{t}{j} \in \mathbb{Z},\ 0 < t \leqslant m-1, \\ 0, & t = 0 \text{ or } \frac{t}{j} \notin \mathbb{Z},\ t \leqslant m-1. \end{cases}$$

The multiplicative structure of $\mathcal{Q}^{(j)}$ *induces a multiplicative structure on the homologies of* $\mathcal{Q}^{(j)}$ *that coincides with the exterior* (*resp. symmetric*) *product in the exterior algebra of* $\operatorname{Ker} d_j$ (*resp. in the divided powers algebra of* $\operatorname{Ker} d_j$) *for* j *odd* (*resp.* j *even*).

Theorem 1 is the principal result in this paper together with the following two corollaries. The proof of Theorem 1 is rather long and is completed in Section 5 after developing some preliminary results in Section 4.

Corollary 1. *Suppose* R *is a subring of* $\mathbb{Q}$ *containing the integers* $\mathbb{Z}$ *and such that every element* $1 \leqslant i \leqslant m-1$ *is invertible in* R *then* $\mathcal{Q}^{(m)}$ *is well-defined and is exact in all dimensions* $i \leqslant m-1$.

Proof. Note that the remarks before the statement of Theorem 1 show that if $(m-1)!$ is invertible in R we can construct $\mathcal{Q}_{j+1}^{(j+1)}$ once we have constructed $\mathcal{Q}_t^{(j)}$ with properties (1) and (2) for $j, t \leqslant m-1$. With other words we can construct $\mathcal{Q}^{(m)}$ and by Theorem 1 $H_s(\mathcal{Q}^{(m)}) = 0$ for $s \leqslant m-1$. □

Corollary 2. *If* R *is the ring of the rationals then the complex* $\mathcal{Q}$ *is well-defined and is exact.*

Proof. Note that $\mathcal{Q} = \bigcup_j \mathcal{Q}^{(j)}$ and hence

$$H_n(\mathcal{Q}) = \varinjlim H_n\bigl(\mathcal{Q}^{(j)}\bigr).$$

By Theorem 1, $H_n(\mathcal{Q}^{(j)}) = 0$ for $j > n$, hence $H_n(\mathcal{Q}) = 0$. □

Finally we note that if M is a module over $\mathbb{Z}[G]$ we can take $\mathcal{A}_j$ to be free G-sets. The differential of the lifted complex $\mathcal{Q}$ commutes with the action of G, where G acts on exterior, divided and tensor products diagonally and on $\mathbb{Z}[M]$ via its action on M.

4. Some exact sequences involving exterior and divided powers

The main result of this section is Proposition 2. We start with some definitions. Suppose V is a free abelian group, we set for $(i, j) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \setminus (0,0)$

$$M_{i,j} = \bigl(\wedge^i V\bigr) \otimes \bigl(\widetilde{S}^j(V)\bigr)$$

and define a $\mathbb{Z}$-linear map $\theta_{i,j} : M_{i,j} \to M_{i+1,j-1}$ for $i \geqslant 0$, $j \geqslant 1$ by

$$\theta_{i,j}\big((v_1\wedge\cdots\wedge v_i)\otimes(w_1*\cdots*w_j)\big)$$
$$=\sum_{1\leqslant t\leqslant j}(v_1\wedge\cdots\wedge v_i\wedge w_t)\otimes\big(w_1*\cdots*\hat{w}_t*\cdots*w_j\big).$$

Fix a linear well ordered basis $V^{(0)}$ of V over $\mathbb{Z}$ and define $M_{i,j}^{(0)}$ to be the set

$$\big\{\lambda(w_1*\cdots*w_j)(v_1\wedge\cdots\wedge v_i)\otimes(w_1*\cdots*w_j)$$
$$\big|\text{ where all } v_k, w_r\in V^{(0)} \text{ and } v_1<\cdots<v_i,\ w_1\leqslant\cdots\leqslant w_j\big\},$$

where $\lambda(w_1*\cdots*w_j)=\frac{1}{\prod_{v\in V^{(0)}}\alpha_v!}$, α_v is the number of the elements of $\{w_1,\dots,w_j\}$ equal to v. Note that $\lambda(w_1*\cdots*w_j)w_1*\cdots*w_j\in\widetilde{S}^j(V)$. We write $\lambda(\underline{w})\underline{v}\otimes\underline{w}$ for the element $\lambda(w_1*\cdots*w_j)(v_1\wedge\cdots\wedge v_i)\otimes(w_1*\cdots*w_j)$ from $M_{i,j}^{(0)}$.

Lemma 1. *$M_{i,j}^{(0)}$ is a basis of $M_{i,j}$ over $\mathbb{Z}$.*

Proof. It is sufficient to show that $M_{0,j}^{(0)}$ is a basis of $M_{0,j}=\widetilde{S}^j(V)$. For $v_1\otimes\cdots\otimes v_j\in\otimes^j V$ define $S(v_1\otimes\cdots\otimes v_j)$ as the symmetrisation of $v_1\otimes\cdots\otimes v_j$, i.e., $S(v_1\otimes\cdots\otimes v_j)=\sum\pi(v_1\otimes\cdots\otimes v_j)$, where the sum is over representatives of the left coset classes $S_j/\mathrm{Stab}_{S_j}(v_1\otimes\cdots\otimes v_j)$. Note that $\widetilde{S}^j(V)$ is spanned over $\mathbb{Z}$ by

$$D=\big\{S(v_1\otimes\cdots\otimes v_j)\mid v_1,\dots,v_j\in V^{(0)},\ v_1\leqslant v_2\leqslant\cdots\leqslant v_j\big\}$$

and $S(v_1\otimes\cdots\otimes v_j)=\lambda(v_1*\cdots*v_j)v_1*\cdots*v_j$. Finally, we show that there is not a non-trivial $\mathbb{Z}$-linear dependence between the elements of D. Suppose that

$$\sum z_{\underline{w}}\lambda(\underline{w})w_1*\cdots*w_j=0,$$

where the sum is over all $w_i\in V^{(0)}$ such that $w_1\leqslant w_2\leqslant\cdots\leqslant w_j$ and $z_{\underline{w}}\in\mathbb{Z}$. Note that if all $w_i\in V^{(0)}$ and $w_1\leqslant w_2\leqslant\cdots\leqslant w_j$ the element $w_1\otimes\cdots\otimes w_j$ appears in the left hand side of the above equation only in the summand $z_{\underline{w}}\lambda(\underline{w})w_1*\cdots*w_j$, hence $z_{\underline{w}}=0$. □

Now we order the elements of $M_{0,j}^{(0)}$ in the following way:

$$\lambda(w_1*\cdots*w_j)(w_1*\cdots*w_j)<\lambda\big(w_1'*\cdots*w_j'\big)\big(w_1'*\cdots*w_j'\big)$$

if and only if there exists a permutation $\sigma\in S_j$ such that $w_{\sigma(i)}\leqslant w_i'$ for all $1\leqslant i\leqslant j$ and for at least one i the inequality is strict. An element $\lambda(\underline{w})\underline{v}\otimes\underline{w}$ of $M_{i,j}^{(0)}$, where $i\geqslant 1$, $j\geqslant 1$, is said to be good if $v_1>w_1$ and $M_{i,j}^{(\mathrm{good})}$ is defined to be the $\mathbb{Z}$-submodule of $M_{i,j}$ spanned by the good elements in $M_{i,j}^{(0)}$. We continue with two easy lemmas. We omit the index of θ when it is clear from the context what it is.

Lemma 2.

(i) *For* $i \geqslant 1$, $j \geqslant 1$ *we have* $\theta(M_{i-1,j+1}) + M_{i,j}^{(\mathrm{good})} = M_{i,j}$;
(ii) *For all* $i \geqslant 1$ *the map* $\theta : M_{i-1,1} \to M_{i,0}$ *is surjective and the map* $\theta : M_{0,i} \to M_{1,i-1}$ *is injective.*

Proof. (i) We show that every element $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ in $M_{i,j}^{(0)}$ can be written as a sum of good elements in $M_{i,j}^{(0)}$ modulo the image of $\theta_{i-1,j+1}$. We do this by induction on $\lambda(\underline{w})\underline{w}$ with respect to the order defined on $M_{0,j}^{(0)}$.

If $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ in $M_{i,j}^{(0)}$ is not good then $v_1 \leqslant w_1$. Then consider

$$\begin{aligned}
&\theta\bigl(\lambda(v_1 * w_1 * \cdots * w_j)(v_1 * w_1 * \cdots * w_j)\bigr)\\
&\quad = \lambda(w_1 * \cdots * w_j)v_1 \otimes (w_1 * \cdots * w_j)\\
&\qquad + \sum_{k\in I} \lambda\bigl(v_1 * w_1 * \cdots * \hat{w}_k * \cdots * w_j\bigr) w_k \otimes \bigl(v_1 * w_1 * \cdots * \hat{w}_k * \cdots * w_j\bigr),
\end{aligned}$$

where I is a subset of $\{1, \ldots, j\}$ such that $\{w_k \mid k \in I\}$ contains all different elements from $\{w_m \mid 1 \leqslant m \leqslant j\} \setminus \{v_1\}$. Thus

$$\begin{aligned}
&\theta\bigl(\lambda(v_1 * w_1 * \cdots * w_j)(v_2 \wedge v_3 \wedge \cdots \wedge v_i) \otimes (v_1 * w_1 * \cdots * w_j)\bigr)\\
&\quad = \lambda(w_1 * \cdots * w_j)(v_2 \wedge v_3 \wedge \cdots \wedge v_i \wedge v_1) \otimes (w_1 * \cdots * w_j)\\
&\qquad + \sum_{k\in I} \lambda\bigl(v_1 * w_1 * \cdots * \hat{w}_k * \cdots * w_j\bigr)(v_2 \wedge v_3 \wedge \cdots \wedge v_i \wedge w_k)\\
&\qquad\qquad \otimes \bigl(v_1 * w_1 * \cdots * \hat{w}_k * \cdots * w_j\bigr).
\end{aligned}$$

This shows that modulo the image of $\theta_{i-1,j+1}$ the element $\lambda(\underline{w})\underline{v} \otimes \underline{w}$ is congruent to $(-1)^i \sum_{k\in I}(v_2 \wedge \cdots \wedge v_i \wedge w_k) \otimes (v_1 * w_1 * \cdots * \hat{w}_k * \cdots * w_j)\lambda(v_1 * w_1 * \cdots * \hat{w}_k * \cdots * w_j)$. After reordering the elements from the exterior parts in the above expression and deleting those for which $w_k \in \{v_2, v_3, \ldots, v_i\}$ we get a sum of elements $\lambda(\underline{w}')\underline{v}' \otimes \underline{w}' \in M_{i,j}^{(0)}$ with coefficients -1 or 1 and $\lambda(\underline{w}')\underline{w}' < \lambda(\underline{w})\underline{w}$.

(ii) The fact that $\theta : M_{i-1,1} \to M_{i,0}$ is surjective is obvious. To prove that $\theta : M_{0,i} \to M_{1,i-1}$ is injective we consider the map $\alpha : M_{1,i-1} \to M_{0,i}$ sending $v \otimes \underline{w}$ to $v * \underline{w}$. Then $\alpha\theta_{0,i} = i.\mathrm{id}_{M_{0,i}}$ and so $\operatorname{Ker}\theta_{0,i}$ is an abelian subgroup of finite exponent in the free abelian group $M_{0,i}$, hence is trivial. □

Lemma 3. *For* $j \geqslant 1$, $i \geqslant 0$ *the quotient* $M_{i+1,j-1}/\theta(M_{i,j})$ *is a free abelian group. Furthermore the image of the good elements in* $M_{i+1,j-1}^{(0)}$ *is a basis of* $M_{i+1,j-1}/\theta(M_{i,j})$ *for* $j \geqslant 2$.

Proof. Observe that $\theta^2 = 0$. For $i \geqslant 1$, $j \geqslant 1$, by Lemma 2,

$$\theta(M_{i,j}) = \theta\bigl(M_{i,j}^{(\mathrm{good})}\bigr) + \theta^2(M_{i-1,j+1}) = \theta\bigl(M_{i,j}^{(\mathrm{good})}\bigr)$$

and applying Lemma 2 again we get

$$M_{i+1,j-1} = \theta(M_{i,j}) + M^{(\mathrm{good})}_{i+1,j-1} = \theta\big(M^{(\mathrm{good})}_{i,j}\big) + M^{(\mathrm{good})}_{i+1,j-1} \quad \text{for } j \geqslant 2,\ i \geqslant 1. \tag{3}$$

From now on we aim to prove that the sum in (3) is direct. It is obvious that we can restrict to the case when V is of finite rank. We show that

$$\big|M^{(0)}_{i+1,j-1}\big| = \mathrm{rk}(M_{i+1,j-1}) = \mu(i,j) + \mu(i+1,j-1) \quad \text{for } j \geqslant 2,\ i \geqslant 1, \tag{4}$$

where $\mu(i,j)$ is the number of good elements in $M^{(0)}_{i,j}$. An element of $M^{(0)}_{i+1,j-1}$ is either good or not good and there is a bijection between the non-good elements of $M^{(0)}_{i+1,j-1}$ and the good elements in $M^{(0)}_{i,j}$. An element $\lambda(w_1 * \cdots * w_{j-1})(v_1 \wedge \cdots \wedge v_{i+1}) \otimes (w_1 * \cdots * w_{j-1})$ of $M^{(0)}_{i+1,j-1}$ is non-good if $v_1 \leqslant w_1$ and it corresponds to the good element

$$\lambda(v_1 * w_1 * \cdots * w_{j-1})(v_2 \wedge \cdots \wedge v_{i+1}) \otimes (v_1 * w_1 * \cdots * w_{j-1}).$$

And the good element $\lambda(w'_1 * \cdots * w'_j)(v'_1 \wedge \cdots \wedge v'_i) \otimes (w'_1 * \cdots * w'_j)$ of $M^{(0)}_{i,j}$ corresponds to the non-good element

$$\lambda\big(w'_2 * \cdots * w'_j\big)\big(w'_1 \wedge v'_1 \wedge \cdots \wedge v'_i\big) \otimes \big(w'_2 * \cdots * w'_j\big) \in M^{(0)}_{i+1,j-1}.$$

By (3), (4) and since $\mathrm{rk}(M^{(\mathrm{good})}_{i,j}) \leqslant \mu(i,j)$ we deduce that both sums in (3) are direct. Hence $M_{i+1,j-1}/\theta(M_{i,j}) \simeq M^{(\mathrm{good})}_{i+1,j-1}$, as required.

If $j \geqslant 2$, $i = 0$ by Lemma 1(i) we have $M_{1,j-1} = \theta(M_{0,j}) + M^{(\mathrm{good})}_{1,j-1}$. The sum is direct if $\mathrm{rk}(M_{1,j-1}) = \mathrm{rk}(M_{0,j}) + \mathrm{rk}(M^{(\mathrm{good})}_{1,j-1})$, which is equivalent to the existence of bijection between the non-good elements of $M^{(0)}_{1,j-1}$ and $M^{(0)}_{0,j}$. This can be done exactly as in the first part of the proof. □

Proposition 1. *Suppose V is a free abelian group.*

(1) *The complex $\mathcal{N}$*

$$0 \to \widetilde{S}^i(V) \to V \otimes \widetilde{S}^{i-1}(V) \to \cdots \to \wedge^{i-2} V \otimes \widetilde{S}^2(V) \to \wedge^{i-1} V \otimes V \to \wedge^i V \to 0$$

with the $\mathbb{Z}$-linear differential that equals the differential θ, i.e.,

$$\begin{aligned}&\partial^{(1)}\big((v_1 \wedge \cdots \wedge v_k) \otimes (w_1 * \cdots * w_{i-k})\big)\\ &\quad = \sum_{1 \leqslant j \leqslant i-k} (v_1 \wedge \cdots \wedge v_k \wedge w_j) \otimes \big(w_1 * \cdots * \hat{w}_j * \cdots * w_{i-k}\big)\end{aligned}$$

is exact.

(2) *The complex* $\mathcal{M}$

$$\begin{aligned}&0 \to \wedge^i(V) \to V \otimes \wedge^{i-1}(V) \to \widetilde{S}^2(V) \otimes \wedge^{i-2}(V) \to \cdots\\ &\quad \to \widetilde{S}^{i-1}(V) \otimes V \to \widetilde{S}^i(V) \to 0\end{aligned}$$

with the $\mathbb{Z}$*-linear differential*

$$\begin{aligned}&\partial^{(2)}\big((w_1 * \cdots * w_k) \otimes (v_1 \wedge \cdots \wedge v_{i-k})\big)\\ &\quad = \sum_{1 \leqslant j \leqslant i-k} (-1)^{j-1}(w_1 * \cdots * w_k * v_j) \otimes \big(v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{i-k}\big)\end{aligned}$$

has homology groups of finite exponent dividing i*. In particular, if* R *is a subring of* $\mathbb{Q}$ *containing the ring of integers such that* i *is invertible in* R *then the complex* $\mathcal{M} \otimes R$ *with differential* $\partial^{(2)} \otimes \mathrm{id}$ *is exact.*

Remark. The result in part (2) of Proposition 1 is the best possible as there are examples when the complex $\mathcal{M}$ is not exact.

Proof. We consider the complex

$$\begin{aligned}&\mathcal{T}: 0 \to \wedge^i(V) \to \wedge^{i-1}(V) \otimes V \to \wedge^{i-2}(V) \otimes \widetilde{S}^2(V) \to \cdots\\ &\quad \to V \otimes \widetilde{S}^{i-1}(V) \to \widetilde{S}^i(V) \to 0\end{aligned}$$

with the $\mathbb{Z}$-linear differential

$$\begin{aligned}&\partial^{(3)}\big((v_1 \wedge \cdots \wedge v_k) \otimes (w_1 * \cdots * w_{i-k})\big)\\ &\quad = \sum_{1 \leqslant j \leqslant k} (-1)^{k-j}\big(v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k\big) \otimes (v_j * w_1 * \cdots * w_{i-k}).\end{aligned}$$

We claim that

$$\partial^{(1)}\partial^{(3)} + \partial^{(3)}\partial^{(1)} = i.\mathrm{id}.$$

Thus the homology groups of $\mathcal{N}$ and the new complex $\mathcal{T}$ are of finite exponent dividing i. By Lemma 3 the homology groups of $\mathcal{N}$ are trivial or $\mathbb{Z}$-torsion-free, hence $\mathcal{N}$ is exact. Furthermore using that $\widetilde{S}^t(V) \otimes \wedge^{i-t}(V) \simeq \wedge^{i-t}(V) \otimes \widetilde{S}^t(V)$ via $\underline{w} \otimes \underline{v} \simeq \underline{v} \otimes \underline{w}$ we see that the complex $\mathcal{M}$ is isomorphic to the complex

$$\begin{aligned}&0 \to \wedge^i(V) \to \wedge^{i-1}(V) \otimes V \to \wedge^{i-2}(V) \otimes \widetilde{S}^2(V) \to \cdots\\ &\quad \to V \otimes \widetilde{S}^{i-1}(V) \to \widetilde{S}^i(V) \to 0\end{aligned}$$

with the $\mathbb{Z}$-linear differential

$$
\begin{aligned}
&\partial^{(4)}\bigl((v_1 \wedge \cdots \wedge v_k) \otimes (w_1 * \cdots * w_{i-k})\bigr)\\
&\quad= \sum_{1 \leqslant j \leqslant k} (-1)^{j-1}\bigl(v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k\bigr) \otimes (v_j * w_1 * \cdots * w_{i-k}).
\end{aligned}
$$

Note that $\partial^{(3)}|_{\wedge^k(V)\otimes\widetilde{S}^{i-k}(V)} = (-1)^{k-1}\partial^{(4)}|_{\wedge^k(V)\otimes\widetilde{S}^{i-k}(V)}$, i.e., the differentials $\partial^{(3)}$ and $\partial^{(4)}$ are equal up to sign. Then the homologies of $\mathcal{M}$ are isomorphic to the homologies of the new complex $\mathcal{T}$, which have been proved to be of finite exponent. Finally we prove the claim.

$$
\begin{aligned}
&\bigl(\partial^{(3)}\partial^{(1)} + \partial^{(1)}\partial^{(3)}\bigr)\bigl((v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k})\bigr)\\
&\quad= \partial^{(3)}\biggl(\sum_{1 \leqslant j \leqslant i-k} (v_1 \wedge \cdots \wedge v_k \wedge w_j) \otimes \bigl(w_1 * \cdots * \hat{w}_j * \cdots * w_{i-k}\bigr)\biggr)\\
&\qquad+ \partial^{(1)}\biggl(\sum_{1 \leqslant \alpha \leqslant k} (-1)^{k-\alpha}\bigl(v_1 \wedge \cdots \wedge \hat{v}_\alpha \wedge \cdots \wedge v_k\bigr) \otimes (v_\alpha * w_1 * \cdots * w_{i-k})\biggr)\\
&\quad= \sum_{1 \leqslant j \leqslant i-k}\ \sum_{1 \leqslant s \leqslant k} (-1)^{k+1-s}\bigl(v_1 \wedge \cdots \wedge \hat{v}_s \wedge \cdots \wedge v_k \wedge w_j\bigr)\\
&\qquad\qquad \otimes \bigl(v_s * w_1 * \cdots * \hat{w}_j * \cdots * w_{i-k}\bigr)\\
&\qquad+ \sum_{1 \leqslant j \leqslant i-k} (v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k})\\
&\qquad+ \sum_{1 \leqslant \alpha \leqslant k} (-1)^{k-\alpha} \sum_{1 \leqslant \beta \leqslant i-k} \bigl(v_1 \wedge \cdots \wedge \hat{v}_\alpha \wedge \cdots \wedge v_k \wedge w_\beta\bigr)\\
&\qquad\qquad \otimes \bigl(v_\alpha * w_1 * \cdots * \hat{w}_\beta * \cdots * w_{i-k}\bigr)\\
&\qquad+ \sum_{1 \leqslant \alpha \leqslant k} (-1)^{k-\alpha}\bigl(v_1 \wedge \cdots \wedge \hat{v}_\alpha \wedge \cdots \wedge v_k \wedge v_\alpha\bigr) \otimes \bigl(w_1 * w_2 * \cdots * w_{i-k}\bigr)\\
&\quad= (i-k)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k}) + k(v_1 \wedge v_2 \wedge \cdots \wedge v_k)\\
&\qquad \otimes (w_1 * w_2 * \cdots * w_{i-k})\\
&\quad= i(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \otimes (w_1 * w_2 * \cdots * w_{i-k}),
\end{aligned}
$$

as required. This completes the proof of the claim and the proposition. □

Proposition 2. *Suppose $d : V \to W$ is a homomorphism of free abelian groups.*

(1) *The following complex denoted by $\mathcal{F}_i(V, d)$*

$$
\begin{aligned}
0 \to \widetilde{S}^i(\operatorname{Ker} d) \xrightarrow{\tau} \widetilde{S}^i(V) \to \operatorname{Im} d \otimes \widetilde{S}^{i-1}(V) \to \cdots\\
\to \wedge^{i-1}(\operatorname{Im} d) \otimes V \to \wedge^i(\operatorname{Im} d) \to 0
\end{aligned}
$$

with the $\mathbb{Z}$-linear differentials

$$\begin{aligned}&\partial_{\mathcal{F}_i}\big((v_1\wedge\cdots\wedge v_k)\otimes(w_1*\cdots*w_{i-k})\big)\\&\quad=\sum_{1\leqslant j\leqslant i-k}\big(v_1\wedge\cdots\wedge v_k\wedge d(w_j)\big)\otimes\big(w_1*\cdots*\hat{w}_j*\cdots*w_{i-k}\big),\end{aligned}$$

where $v_1,\dots,v_k\in\operatorname{Im}d, w_1,\dots,w_{i-k}\in V$ *is exact. Here* τ *is the natural embedding induced by the embedding of* $\operatorname{Ker}d$ *in* V.

(2) *Let* $\mathcal{R}_i(V,d)$ *be the complex*

$$\begin{aligned}&0\to\wedge^i(\operatorname{Ker}d)\xrightarrow{\tau}\wedge^i(V)\to(\operatorname{Im}d)\otimes\wedge^{i-1}(V)\to\widetilde{S}^2(\operatorname{Im}d)\otimes\wedge^{i-2}(V)\to\cdots\\&\quad\to\widetilde{S}^{i-1}(\operatorname{Im}d)\otimes V\to\widetilde{S}^i(\operatorname{Im}d)\to 0\end{aligned}$$

with the $\mathbb{Z}$*-linear differential*

$$\begin{aligned}&\partial_{\mathcal{R}_i}\big((w_1*\cdots*w_k)\otimes(v_1\wedge\cdots\wedge v_{i-k})\big)\\&\quad=\sum_{1\leqslant j\leqslant i-k}(-1)^{j-1}\big(w_1*\cdots*w_k*d(v_j)\big)\otimes\big(v_1\wedge\cdots\wedge\hat{v}_j\wedge\cdots\wedge v_{i-k}\big),\end{aligned}$$

where $w_1,\dots,w_k\in\operatorname{Im}d, v_1,\dots,v_{i-k}\in V$ *and* τ *is the embedding induced by the embedding of* $\operatorname{Ker}d$ *in* V. *Then* $\mathcal{R}_i(V,d)$ *has homology groups of finite exponent not bigger than* i. *In particular if* R *is a subring of* $\mathbb{Q}$ *containing the ring of the integers such that* $i!$ *is invertible in* R *then* $\mathcal{R}_i(V,d)\otimes R$ *is exact.*

Proof. If d is injective we identify V with its image $\operatorname{Im}d$ and apply Proposition 1.

If d is not injective we define V_1 to be the quotient $V/\operatorname{Ker}d$ and $\hat{d}: V_1\to W$ the map induced by d. As W is free abelian its subgroups are free abelian, in particularly $\hat{d}(V_1)\simeq V_1$ is free abelian. Thus $V\simeq V_1\oplus(\operatorname{Ker}d)$ as abelian groups. Using the fact that

$$\widetilde{S}^k(V)\simeq\widetilde{S}^k\big(V_1\oplus(\operatorname{Ker}d)\big)\simeq\bigoplus_{k_1+k_2=k}\big(\widetilde{S}^{k_1}(V_1)\otimes\widetilde{S}^{k_2}(\operatorname{Ker}d)\big)$$

(where the tensor product in the right-hand side of the formula corresponds to the symmetric product $*$ on the left-hand side), we see that the complex $\mathcal{F}_i(V,d)$ splits into a direct sum of the complexes

$$\big\{\mathcal{F}_j\big(V_1,\hat{d}\big)\otimes\widetilde{S}^{i-j}(\operatorname{Ker}d)\big\}_{1\leqslant j\leqslant i}\quad\text{with differentials }\partial_{\mathcal{F}_j}\otimes\mathrm{id}_{\widetilde{S}^{i-j}}$$

and the complex

$$0\to\widetilde{S}^i(\operatorname{Ker}d)\xrightarrow{\mathrm{id}}\widetilde{S}^i(\operatorname{Ker}d)\to 0.$$

By Proposition 1(1), $\mathcal{F}_j(V_1,\hat{d})$ is exact and hence $\mathcal{F}_j(V_1,\hat{d})\otimes\widetilde{S}^{i-j}(\operatorname{Ker}d)$ and $\mathcal{F}_i(V,d)$ are exact.

The proof of the second part of Proposition 2 is similar. Using

$$\wedge^j\big(V_1 \oplus (\operatorname{Ker} d)\big) \simeq \bigoplus_{j_1+j_2=j} \big(\wedge^{j_1}(V_1) \otimes \wedge^{j_2}(\operatorname{Ker} d)\big)$$

(where the tensor product in the right hand side of the formula corresponds to the wedge product $\wedge$ on the left-hand side), we see that $\mathcal{R}_i(V,d)$ splits as a direct sum of complexes

$$\big\{\mathcal{R}_j\big(V_1,\hat{d}\big) \otimes \wedge^{i-j}(\operatorname{Ker} d)\big\}_{1\leqslant j\leqslant i} \quad \text{with the differential } \partial_{\mathcal{R}_j} \otimes \mathrm{id}_{\wedge^{i-j}},$$

and the complex

$$0 \to \wedge^i(\operatorname{Ker} d) \xrightarrow{\mathrm{id}} \wedge^i(\operatorname{Ker} d) \to 0.$$

By Proposition 1(2), $\mathcal{R}_j(V_1,\hat{d})$ has homology groups of finite exponent dividing j and as $\wedge^{i-j}(\operatorname{Ker} d)$ is a free $\mathbb{Z}$-module $H_*(\mathcal{R}_j(V_1,\hat{d}) \otimes \wedge^{i-j}(\operatorname{Ker} d)) = H_*(\mathcal{R}_j(V_1,\hat{d})) \otimes \wedge^{i-j}(\operatorname{Ker} d)$. This completes the proof. □

5. Proof of Theorem 1

We prove Theorem 1 by induction on j. Consider first the case $j=1$ and denote by A_1 the free abelian group $\mathbb{Z}\mathcal{A}_1$. Let $\mathcal{P}$ be the Koszul resolution of $\mathbb{Z}$ over $\mathbb{Z}[A_1]$ constructed using the basis $\mathcal{A}_1$, i.e., $P_i = \mathbb{Z}[A_1] \otimes (\wedge^i A_1)$ with differentials

$$\partial(a_1 \wedge \cdots \wedge a_i) = \sum_{1\leqslant j\leqslant i} (-1)^{j-1}(a_j - 1) \otimes a_1 \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_i$$

for $a_1 < \cdots < a_i \in \mathcal{A}_1$. We consider $\mathcal{P}$ as a free resolution of the trivial module $\mathbb{Z}$ over $\mathbb{Z}[\operatorname{Ker} d_1]$ and use this resolution to calculate $H_i(\operatorname{Ker} d_1, \mathbb{Z})$. By the description of the homologies of torsion free abelian group as exterior powers given in Section 1.2

$$H_i(\mathbb{Z} \otimes_{\mathbb{Z}(\operatorname{Ker} d_1)} \mathcal{P}) = H_i(\operatorname{Ker} d_1, \mathbb{Z}) \simeq \wedge^i \operatorname{Ker} d_1.$$

Remember that by definition

$$\mathcal{Q}^{(1)} \simeq (\mathbb{Z} \otimes_{\mathbb{Z}(\operatorname{Ker} d_1)} \mathcal{P}) \otimes_{\mathbb{Z}} R.$$

As R is flat over $\mathbb{Z}$

$$H_i\big(\mathcal{Q}^{(1)}\big) = H_i(\mathbb{Z} \otimes_{\mathbb{Z}(\operatorname{Ker} d_1)} \mathcal{P}) \otimes R \simeq \big(\wedge^i \operatorname{Ker} d_1\big) \otimes R,$$

thus Theorem 1 holds for $j = 1$. Note that at this stage we have not used the fact that some of the elements of R are invertible. This fact will be used later on when we refer to Proposition 2(2). We remind the reader that by construction for $j \geqslant 2$

$$\mathcal{Q}_m^{(j)} = \begin{cases} \bigoplus_{m_1+j.m_2=m} \mathcal{Q}_{m_1}^{(j-1)} \otimes \wedge^{m_2}(\mathbb{Z}\mathcal{A}_j) & \text{if } j \text{ is odd,} \\ \bigoplus_{m_1+j.m_2=m} \mathcal{Q}_{m_1}^{(j-1)} \otimes \widetilde{S}^{m_2}(\mathbb{Z}\mathcal{A}_j) & \text{if } j \text{ is even} \end{cases} \tag{5}$$

with differential

$$\partial(\mathbb{Z}\mathcal{A}_j) \subseteq \mathcal{Q}_{j-1}^{(j-1)} \cap \operatorname{Ker} \partial, \quad \partial|_{\mathbb{Z}\mathcal{A}_j} \equiv d_j \quad \text{modulo } \partial\big(\mathcal{Q}^{(j-1)}\big).$$

Note that by (1) we have the following more explicit formulas for the differentials. If $j \geqslant 2$ is even

$$\begin{aligned} \partial\big(\lambda \otimes (w_1 * \cdots * w_{m_2})\big) = {} & \partial(\lambda) \otimes (w_1 * \cdots * w_{m_2}) \\ & + (-1)^{m_1} \sum_{1 \leqslant i \leqslant m_2} \lambda w_1 \cdots \partial(w_i) \cdots w_{m_2}, \end{aligned} \tag{6}$$

where $\lambda \in \mathcal{Q}_{m_1}^{(j-1)}$, $w_1, \ldots, w_{m_2} \in \mathbb{Z}\mathcal{A}_j$, $\lambda w_1 \cdots \partial(w_i) \cdots w_{m_2}$ is the product of the corresponding elements in $\mathcal{Q}^{(j)}$.

If $j \geqslant 3$ is odd

$$\begin{aligned} \partial\big(\lambda \otimes (v_1 \wedge \cdots \wedge v_{m_2})\big) = {} & \partial(\lambda) \otimes (v_1 \wedge \cdots \wedge v_{m_2}) \\ & + \sum_{1 \leqslant i \leqslant m_2} (-1)^{i-1+m_1} \lambda v_1 \cdots \partial(v_i) \cdots v_{m_2} \end{aligned} \tag{7}$$

for $\lambda \in \mathcal{Q}_{m_1}^{(j-1)}$, $v_1, \ldots, v_{m_2} \in \mathbb{Z}\mathcal{A}_j$.

Now we consider the inductive step for $j \geqslant 2$. Assume Theorem 1 holds for the complex $\mathcal{Q}^{(j-1)}$ and we consider the filtration $\{\mathcal{F}^p\}_{p \geqslant 1}$ of $\mathcal{Q}^{(j)}$ where $\mathcal{F}^p$ contains all direct components of $\mathcal{Q}^{(j)}$ in (5) with $m_2 \leqslant p$. Thus

$$\begin{aligned} \big(\mathcal{F}^p/\mathcal{F}^{p-1}\big)_s &\simeq \mathcal{Q}_{s-pj}^{(j-1)} \otimes \wedge^p(\mathbb{Z}\mathcal{A}_j) \quad \text{for } j \text{ odd;} \\ \big(\mathcal{F}^p/\mathcal{F}^{p-1}\big)_s &\simeq \mathcal{Q}_{s-pj}^{(j-1)} \otimes \widetilde{S}^p(\mathbb{Z}\mathcal{A}_j) \quad \text{for } j \text{ even;} \end{aligned}$$

with differentials $\partial|_{\mathcal{Q}^{(j-1)}} \otimes \mathrm{id}$ by (1). The associated spectral sequence as defined in [8, Corollary 11.12] is

$$E_{p,q}^1 = H_{p+q}\big(\mathcal{F}^p/\mathcal{F}^{p-1}\big)$$

with differential

$$d_{p,q}^1 : E_{p,q}^1 \to E_{p-1,q}^1,$$

which by [8, Exercise 11.13] is the connecting homomorphism of the exact sequence of complexes $0 \to \mathcal{F}^{p-1}/\mathcal{F}^{p-2} \to \mathcal{F}^p/\mathcal{F}^{p-2} \to \mathcal{F}^p/\mathcal{F}^{p-1} \to 0$.

(I) Now we assume that $j \geqslant 2$ is odd. Using the description of the differential of $\mathcal{F}^p/\mathcal{F}^{p-1}$ and assumption that the inductive hypothesis holds for $j-1$ we get

$$E^1_{p,q} \simeq \begin{cases} H_{p+q-jp}(\mathcal{Q}^{(j-1)}) \otimes \wedge^p(\mathbb{Z}\mathcal{A}_j) & \text{if } p+q-jp > 0,\\ R \otimes \wedge^p(\mathbb{Z}\mathcal{A}_j) & \text{if } p+q-jp = 0,\\ 0 & \text{if } p+q-jp < 0, \end{cases}$$

$$\simeq \begin{cases} \text{not known groups} & \text{if } p+q-jp \geqslant m,\\ \widetilde{S}^{\frac{p+q-jp}{j-1}}(\operatorname{Ker} d_{j-1}) \otimes \wedge^p(\mathbb{Z}\mathcal{A}_j) \otimes R & \text{if } \frac{q}{j-1} \in \mathbb{Z},\ 0 \leqslant p+q-jp \leqslant m-1,\\ 0 & \text{if } \frac{q}{j-1} \notin \mathbb{Z},\ 0 \leqslant p+q-jp \leqslant m-1,\\ 0 & \text{if } p+q-jp < 0. \end{cases}$$

Claim. *After substituting* $\operatorname{Ker} d_{j-1}$ *with* $\operatorname{Im} d_j$ *in the above formula the differential of* $E^1_{p,q}$ *for* $\frac{q}{j-1} \in \mathbb{Z}$, $0 \leqslant p+q-jp \leqslant m-1$ *is* $\tilde{\partial} \otimes \mathrm{id}_R$, *where* $\tilde{\partial}$ *up to sign is the differential defined in Proposition* 2(2) *for* $d = d_j : V = \mathbb{Z}\mathcal{A}_j \to W = \mathbb{Z}\mathcal{A}_{j-1}$.

Proof. We use the description of $d^1_{p,q}$ as connecting homomorphism. Let $r = \sum_i \lambda_i \otimes v^{(i)}$ be an element of $\mathcal{Q}^{(j-1)}_{s-pj} \otimes \wedge^p \mathbb{Z}\mathcal{A}_j \subset \mathcal{F}^p$, where $\lambda_i \in \mathcal{Q}^{(j-1)}_{s-pj}$, $v^{(i)} = v^{(i)}_1 \wedge v^{(i)}_2 \wedge \cdots \wedge v^{(i)}_p$ is an element of a fixed basis of $\wedge^p \mathbb{Z}\mathcal{A}_j$ over $\mathbb{Z}$. We assume that the image of r in $\mathcal{F}^p/\mathcal{F}^{p-1}$ belongs to $\operatorname{Ker}(\partial_{\mathcal{F}^p/\mathcal{F}^{p-1}})$. Thus $\partial(r) \in \mathcal{F}^{p-1}$. By (7) we have $\partial(r) \in \sum_i \partial(\lambda_i) \otimes v^{(i)} + \mathcal{F}^{p-1}$, hence $\partial(\lambda_i) = 0$ for all i. Then by the description of d^1 as connecting homomorphism we have for the class $[r] \in H_m(\mathcal{F}^p/\mathcal{F}^{p-1})$ of r that

$$d^1\big([r]\big) = \big[\partial(r)\big] \in H_{m-1}\big(\mathcal{F}^{p-1}/\mathcal{F}^{p-2}\big).$$

Since $\partial(\lambda_i) = 0$, $\deg(\lambda_i) = s - pj$ and by (7)

$$\begin{aligned} \partial(r) &= \sum_i \Bigg(\partial(\lambda_i) \otimes v^{(i)} + \lambda_i \sum_{t \leqslant p} (-1)^{t-1+\deg(\lambda_i)} v^{(i)}_1 \cdots v^{(i)}_{t-1} \partial\big(v^{(i)}_t\big) v^{(i)}_{t+1} \cdots v^{(i)}_p \Bigg) \\ &= \sum_i \Bigg(\lambda_i \sum_{t \leqslant p} (-1)^{t-1+s-pj} v^{(i)}_1 \cdots v^{(i)}_{t-1} \partial\big(v^{(i)}_t\big) v^{(i)}_{t+1} \cdots v^{(i)}_p \Bigg). \end{aligned} \tag{8}$$

Furthermore by (2) there exists $\mu^{(i)}_t \in \mathcal{Q}^{(j-1)}_j$ such that

$$\partial\big(v^{(i)}_t\big) - d_j\big(v^{(i)}_t\big) = \partial\big(\mu^{(i)}_t\big). \tag{9}$$

We aim to prove that for every $t \leqslant p$

$$\lambda_i v^{(i)}_1 \cdots v^{(i)}_{t-1} \partial\big(\mu^{(i)}_t\big) v^{(i)}_{t+1} \cdots v^{(i)}_p \in \mathcal{F}^{p-2} + \partial\big(\mathcal{F}^{p-1}\big). \tag{10}$$

Indeed

$$\begin{aligned}
&\partial\big(\lambda_i v_1^{(i)}\cdots v_{t-1}^{(i)}\mu_t^{(i)}v_{t+1}^{(i)}\cdots v_p^{(i)}\big)\\
&\quad=\partial(\lambda_i)v_1^{(i)}\cdots v_{t-1}^{(i)}\mu_t^{(i)}v_{t+1}^{(i)}\cdots v_p^{(i)}+(-1)^{\deg(\lambda_i)}\lambda_i\partial\big(v_1^{(i)}\cdots v_{t-1}^{(i)}\mu_t^{(i)}v_{t+1}^{(i)}\cdots v_p^{(i)}\big)\\
&\quad=(-1)^{\deg(\lambda_i)}\lambda_i\partial\big(v_1^{(i)}\cdots v_{t-1}^{(i)}\mu_t^{(i)}v_{t+1}^{(i)}\cdots v_p^{(i)}\big)\\
&\quad=(-1)^{\deg(\lambda_i)}\lambda_i\bigg(\sum_{k<t}(-1)^{k-1}v_1^{(i)}\cdots\partial\big(v_k^{(i)}\big)\cdots\mu_t^{(i)}\cdots v_p^{(i)}\\
&\qquad\qquad+\sum_{k>t}(-1)^{k-1}v_1^{(i)}\cdots\mu_t^{(i)}\cdots\partial\big(v_k^{(i)}\big)\cdots v_p^{(i)}\\
&\qquad\qquad+(-1)^{t-1}v_1^{(i)}\cdots\partial\big(\mu_t^{(i)}\big)\cdots v_p^{(i)}\bigg)\\
&\quad\in(-1)^{\deg(\lambda_i)+t-1}\lambda_i v_1^{(i)}\cdots\partial(\mu_t^{(i)})\cdots v_p^{(i)}+\mathcal{F}^{p-2}.
\end{aligned}$$

Note that the above calculation implies (10). Then by (8) and (9)

$$\begin{aligned}
\big[\partial(r)\big]&=\bigg[\sum_i\bigg(\lambda_i\sum_{t\leqslant p}(-1)^{t-1+s-pj}v_1^{(i)}\cdots v_{t-1}^{(i)}\big(\partial\big(\mu_t^{(i)}\big)+d_j\big(v_t^{(i)}\big)\big)v_{t+1}^{(i)}\cdots v_p^{(i)}\bigg)\bigg]\\
&=\bigg[\sum_i\bigg(\lambda_i\sum_{t\leqslant p}(-1)^{t-1+s-pj}v_1^{(i)}\cdots v_{t-1}^{(i)}\partial\big(\mu_t^{(i)}\big)v_{t+1}^{(i)}\cdots v_p^{(i)}\bigg)\bigg]\\
&\quad+\bigg[\sum_i\bigg(\lambda_i\sum_{t\leqslant p}(-1)^{t-1+s-pj}v_1^{(i)}\cdots v_{t-1}^{(i)}d_j\big(v_t^{(i)}\big)v_{t+1}^{(i)}\cdots v_p^{(i)}\bigg)\bigg]\\
&\in H_{m-1}\big(\mathcal{F}^{p-1}/\mathcal{F}^{p-2}\big).
\end{aligned}$$

By (10)

$$\bigg[\lambda_i\sum_{t\leqslant p}(-1)^{t-1+s-pj}v_1^{(i)}\cdots v_{t-1}^{(i)}\partial\big(\mu_t^{(i)}\big)v_{t+1}^{(i)}\cdots v_p^{(i)}\bigg]=0,$$

hence

$$\big[\partial(r)\big]=\bigg[\sum_i\bigg(\lambda_i\sum_{t\leqslant p}(-1)^{t-1+s-pj}v_1^{(i)}\cdots v_{t-1}^{(i)}d_j\big(v_t^{(i)}\big)v_{t+1}^{(i)}\cdots v_p^{(i)}\bigg)\bigg].$$

Finally $d_j(v_t^{(i)})\in\mathbb{Z}\mathcal{A}_{j-1}$ is a central element in $\mathcal{Q}$ (remember $j-1$ is even), hence

$$\begin{aligned}
\big[\partial(r)\big]&=(-1)^{s-pj}\bigg[\sum_i\bigg(\sum_{t\leqslant p}(-1)^{t-1}\lambda_i d_j\big(v_t^{(i)}\big)v_1^{(i)}\cdots v_{t-1}^{(i)}v_{t+1}^{(i)}\cdots v_p^{(i)}\bigg)\bigg]\\
&=(-1)^{s-pj}\sum_i\sum_{t\leqslant p}(-1)^{t-1}\big[\lambda_i d_j\big(v_t^{(i)}\big)\big]\otimes v_1^{(i)}\cdots v_{t-1}^{(i)}v_{t+1}^{(i)}\cdots v_p^{(i)}.
\end{aligned}$$

The latest equality comes from the fact that the differential of $\mathcal{F}^{p-1}/\mathcal{F}^{p-2}$ is $\partial \otimes \mathrm{id}_{\wedge^{p-1}}$. This completes the proof of the claim. □

Then the bigraded module $\{E^1_{p,q}\}_{p+q-jp \leqslant m-1}$ splits into chains of complexes:

$$\begin{aligned} &0 \to \wedge^p(V) \otimes R \to (\operatorname{Im} d) \otimes \wedge^{p-1}(V) \otimes R \to \widetilde{S}^2(\operatorname{Im} d) \otimes \wedge^{p-2}(V) \otimes R \to \cdots \\ &\quad \to \widetilde{S}^{p-1}(\operatorname{Im} d) \otimes V \otimes R \to \widetilde{S}^p(\operatorname{Im} d) \otimes R \to 0 \end{aligned}$$

with differentials up to sign the ones given in Proposition 2(2) for $d = d_j : V = \mathbb{Z}\mathcal{A}_j \to \mathbb{Z}\mathcal{A}_{j-1}$, the elements of V have degree j and the elements of $\operatorname{Im} d$ degree $j-1$. According to Proposition 2(2) these complexes are exact in all dimensions except at the very beginning, where the homology is isomorphic to $\wedge^p(\operatorname{Ker} d_j) \otimes R$, i.e.,

$$E^2_{p,q} = \begin{cases} \text{not known} & \text{if } p+q-jp \geqslant m, \\ \wedge^p(\operatorname{Ker} d_j) \otimes R & \text{if } p+q-jp = 0, \\ 0 & \text{if } p+q-jp \in (-\infty, m-1] \setminus \{0\}. \end{cases}$$

If $d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$ is non-trivial for some $p+q \leqslant m-1$ then $E^2_{p,q} \neq 0 \neq E^2_{p-2,q+1}$. Since $(p-2)+(q+1)-j(p-2) \leqslant m-1$ by the above description of $E^2_{p,q}$ we have $\frac{q}{p} = j-1 = \frac{q+1}{p-2} > \frac{q}{p}$, a contradiction. Thus $E^2_{p,q} = E^3_{p,q}$ and the same argument shows $E^3_{p,q} = E^4_{p,q} = \cdots = E^\infty_{p,q}$ for $p+q \leqslant m-1$. As

$$E^2_{p,q} \Longrightarrow H_n\bigl(\mathcal{Q}^{(j)}\bigr),$$

we get for $1 \leqslant n \leqslant m-1$

$$H_n\bigl(\mathcal{Q}^{(j)}\bigr) = \begin{cases} \wedge^{\frac{n}{j}}(\operatorname{Ker} d_j) \otimes R & \text{if } \frac{n}{j} \in \mathbb{Z}, \\ 0 & \text{if } \frac{n}{j} \notin \mathbb{Z}. \end{cases}$$

This completes the inductive case when j is odd.

(II) Now we assume that j is even, this case is similar to the previous one. Using the inductive hypothesis

$$\begin{aligned} E^1_{p,q} &\simeq \begin{cases} H_{p+q-jp}(\mathcal{Q}^{(j-1)}) \otimes \widetilde{S}^p(\mathbb{Z}\mathcal{A}_j) & \text{if } p+q-jp > 0, \\ \widetilde{S}^p(\mathbb{Z}\mathcal{A}_j) \otimes R & \text{if } p+q-jp = 0, \\ 0 & \text{if } p+q-jp < 0, \end{cases} \\ &\simeq \begin{cases} \text{not known groups} & \text{if } p+q-jp \geqslant m, \\ \wedge^{\frac{p+q-jp}{j-1}}(\operatorname{Ker} d_{j-1}) \otimes \widetilde{S}^p(\mathbb{Z}\mathcal{A}_j) \otimes R & \text{if } \frac{q}{j-1} \in \mathbb{Z}, 0 \leqslant p+q-jp \leqslant m-1, \\ 0 & \text{if } \frac{q}{j-1} \notin \mathbb{Z}, 0 \leqslant p+q-jp \leqslant m-1, \\ 0 & \text{if } p+q-jp < 0. \end{cases} \end{aligned}$$

We substitute $\operatorname{Ker} d_{j-1}$ with $\operatorname{Im} d_j$ in the above formula and similarly to the previous case E^1 has differentials $\tilde{\partial} \otimes \mathrm{id}_R$, where $\tilde{\partial}$ up to sign is the differential of $\mathcal{F}_i(V, d)$ given in

Proposition 2(1) for $d = d_j : V = \mathbb{Z}\mathcal{A}_j \to W = \mathbb{Z}\mathcal{A}_{j-1}$ (this description of the differential follows from (6) in exactly the same way as the proof of the claim in the previous case follows from (7)). Then the bigraded module $\{E^1_{p,q}\}_{p+q-jp\leqslant m-1}$ splits into chains of complexes:

$$\begin{aligned} 0 &\to \widetilde{S}^p(V)\otimes R \to \operatorname{Im} d_j \otimes \widetilde{S}^{p-1}(V)\otimes R \to \cdots \\ &\to \wedge^{p-1}(\operatorname{Im} d_j)\otimes V\otimes R \to \wedge^p(\operatorname{Im} d_j)\otimes R \to 0, \end{aligned}$$

where $d = d_j : V = \mathbb{Z}\mathcal{A}_j \to \mathbb{Z}\mathcal{A}_{j-1}$ and with differential up to sign given by Proposition 2(1). According to Proposition 2(1) these complexes are exact in all dimensions except at the very beginning, i.e., for $p+q-jp \leqslant m-1$

$$E^2_{p,q} = \begin{cases} \widetilde{S}^p(\operatorname{Ker} d_j)\otimes R & \text{if } p+q-jp = 0, \\ 0 & \text{if } p+q-jp \in (-\infty, m-1]\setminus\{0\}. \end{cases}$$

As before examining the bidegree of the differential d^r of E^r we obtain that $E^\infty_{p,q} = E^2_{p,q}$ for $p+q \leqslant m-1$. This completes the inductive case when j is even as the convergence of the spectral sequence

$$E^2_{p,q} \Longrightarrow H_n\big(\mathcal{Q}^{(j)}\big)$$

implies for $1 \leqslant n \leqslant m-1$

$$H_n\big(\mathcal{Q}^{(j)}\big) = \begin{cases} \widetilde{S}^{\frac{n}{j}}(\operatorname{Ker} d_j)\otimes R & \text{if } \frac{n}{j} \in \mathbb{Z}, \\ 0 & \text{if } \frac{n}{j} \notin \mathbb{Z}. \end{cases}$$

6. Lifting a resolution of M of length 2

Suppose $\mathcal{A}_i = \emptyset$ for $i \geqslant 3$ and let us consider the complex $\mathcal{Q}$ defined in Section 2 for $R = \mathbb{Z}$. Then $\mathcal{Q} = \mathcal{Q}^{(2)}$. As shown at the beginning of Section 4

$$H_*\big(\mathcal{Q}^{(1)}\big) = \wedge^* \operatorname{Ker} d_1.$$

Using the spectral sequence argument from Section 5 for $R = \mathbb{Z}$ and $j = 2$ we get

$$E^1_{p,q} = \begin{cases} \wedge^{q-p}(\operatorname{Ker} d_1)\otimes \widetilde{S}^p(\mathbb{Z}\mathcal{A}_2) & \text{if } q \geqslant p, \\ 0 & \text{if } q < p. \end{cases}$$

We substitute $\operatorname{Ker} d_1$ with $\operatorname{Im} d_2$ in the above formula and note that the differentials of E^1 up to a sign are given by the differentials of $\mathcal{F}_i(V, d)$ in Proposition 2(1), for $d = d_2 : V = \mathbb{Z}\mathcal{A}_2 \to \mathbb{Z}\mathcal{A}_1$. It is important that the first part of Proposition 2 does not require that some of the elements of R are invertible and we could apply it for $R = \mathbb{Z}$.

As $\operatorname{Ker} d_2 = 0$ this gives that (E^1, d^1) splits into exact sequences, the spectral sequence collapses and hence

$$E^2 = 0.$$

In particular, $\mathcal{Q}$ is exact in all dimensions and we obtain the following result.

Theorem 2. *If $\mathcal{A}_i = \emptyset$ for $i \geqslant 3$ then the complex $\mathcal{Q}$ for $R = \mathbb{Z}$ is exact in all dimensions.*

We can use the resolution $\mathcal{Q}$ to calculate the homology groups $H_*(M, \mathbb{Z})$. The complex $\mathbb{Z} \otimes_{\mathbb{Z}[M]} \mathcal{Q}$ splits into direct sum of the subcomplexes

$$\mathcal{B}_t(d_2): \cdots \to \wedge^i(\mathbb{Z}\mathcal{A}_1) \otimes \widetilde{S}^{t-i}(\mathbb{Z}\mathcal{A}_2) \to \wedge^{i+1}(\mathbb{Z}\mathcal{A}_1) \otimes \widetilde{S}^{t-i-1}(\mathbb{Z}\mathcal{A}_2) \to \cdots$$

with differential as in Proposition 2(1) and $\wedge^i(\mathbb{Z}\mathcal{A}_1) \otimes \widetilde{S}^{t-i}(\mathbb{Z}\mathcal{A}_2)$ placed in degree $i + 2(t - i)$. Thus

$$H_n(M, \mathbb{Z}) \simeq \bigoplus_{i+2j=n} H_n\big(\mathcal{B}_{i+j}(d_2)\big)$$

but the isomorphism is not functorial and depends on the choice of exact sequence $0 \to \mathbb{Z}\mathcal{A}_2 \to \mathbb{Z}\mathcal{A}_1 \to M \to 0$ of $\mathbb{Z}$-modules. Note that

$$H_n\big(\mathcal{B}_n(d_2)\big) \simeq \wedge^n M,$$

which is not a surprise as $\wedge^n M$ naturally embeds in $H_n(M, \mathbb{Z})$ as shown in [5, Theorem 6.4], [2, Proposition 5.1]. We summarise

Corollary 3. *If $0 \to \mathbb{Z}\mathcal{A}_2 \xrightarrow{d_2} \mathbb{Z}\mathcal{A}_1 \to M \to 0$ is an exact sequence of abelian groups then there is a non-functorial isomorphism*

$$H_n(M, \mathbb{Z}) \simeq \bigoplus_{0 \leqslant j \leqslant [\frac{n}{2}]} H_n\big(\mathcal{B}_{n-j}(d_2)\big).$$

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