

A CANONICAL COUNTRYMAN LINE

by

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DEDICATION

To Dillon

ABSTRACT

The objective of this thesis is to give a novel proof that the uncountable linear order $C(\rho_0)$ is a Countryman line using an innovative presentation of the full lower trace from Todorcevic's method of minimal walks. In order to motivate the subject we will explicate the basis problem for the class of uncountable linear orders and prove several facts indicating the intrinsic interest of Countryman lines. In order to facilitate the proof we will give a short primer on the method of minimal walks.

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1 PRELIMINARIES

1.1 Chapter Outline

Chapter 2 is devoted to the background, definition, and properties of the uncountable linear orders known as Countryman lines. Particular note will be taken of the canonical aspect of these lines and their relation to the basis problem for the class of uncountable linear orders. Chapter 3 is a short primer on Todorćević's method of minimal walks. Of interest will be a new and interesting presentation of Todorćević's full lower trace, possessing the same properties but with a closed form and clearer definition. Chapter 4 discusses previous constructions of Countryman lines that have appeared in print and then uses the full lower trace and other objects and facts from the method of minimal walks to prove that the uncountable linear order $C(\rho_0)$ is Countryman.

1.2 Background

Throughout this thesis we work in ZFC, the system of Zermelo-Fraenkel axioms with the Axiom of Choice. We use standard notation for standard set theoretic objects and those terms that we leave undefined can be found in any standard reference for set theory, for example Jech [7] or Kunen [9]. For more information on Countryman lines the reader is referred to Bekkali [4], Shelah [13], and Todorćević [16], [18] and [21]. The standard reference for the method of minimal walks is Todorćević's [21].

Since we are working in the space of countable ordinals we have been able to give closed form definitions for many of the functions associated with the method of minimal walks and have subsequently been able to reduce the instances of induction to standard induction over the natural numbers. Hence, this thesis can be read profitably by anyone possessing a strong familiarity with abstract arguments who has taken an upper division undergraduate course in set theory, say chapters 1-4, 6 and 9 of Hrbacek and Jech's [6].

2 COUNTRYMAN LINES

2.1 Introduction

All mathematicians are familiar with the uncountable linear order $(\mathbb{R}, <)$, which is, up to isomorphism, the unique dense, complete, unbounded linear order *including a countable dense subset*. Many mathematicians are also familiar with the uncountable linear order (ω_1, \in) , which is, up to isomorphism, the unique uncountable well-ordered linear order for which *every initial segment is countable*. However, few mathematicians are aware of the uncountable linear orders called Countryman lines, which are characterized by the fact that their *Cartesian square is the union of countably many chains* under the product order.

Countryman orders were first isolated in a short paper written by Roger Countryman in 1971 [5]. The following quote is taken from the paper.

In the proof of theorem 5, the essence of the contradiction obtained was that a certain linearly ordered set had to have the property that its square under the product partial order was the union of a countable number of chains. That this was impossible followed from the fact that the set was uncountable and contained a countable order dense subset. It seems to the author that the existence of a countable order dense subset should not really be needed and the following conjecture is very tempting (and frustrating to try and prove).

CONJECTURE. Let X be a linearly ordered set of cardinal α . Let $X \times X$ be partially ordered by the product order. Then $X \times X$ is not the union of fewer than α subsets each of whose induced order is linear.

The reason Countryman found the conjecture frustrating to prove is because it is false, which was first shown by Shelah [13]. Not only do they exist, in some sense Countryman lines are a canonical uncountable linear order in that it is consistent with the usual axioms of set theory that a basis for the class of uncountable linear orderings contains a Countryman line and its converse. This is the subject of Section 2.3. Section 2.4 then gives several facts about Countryman lines that show why these uncountable linear orders are of further interest.

2.2 Notation and Definitions

In order to facilitate the discussion in Section 2.3 and the proofs in Section 2.4 we give some notation and definitions.

If $(A, <)$ is a (linearly) ordered set we denote the order type of $(A, <)$ by $\text{tp}(A, <)$ and the converse of $(A, <)$ by $(A, <^*) = (A, >)$, i.e. for all $a, a' \in A$,

$$a < a' \text{ if and only if } a' <^* a.$$

When talking about a particular order $(A, <)$ and its converse we will often omit the relation sign and just speak of A and A^* . Likewise, if $\text{tp}(A, <) = \varphi$ then we denote $\text{tp}(A, <^*)$ by φ^* . Following Cantor we will let $\text{tp}(\mathbb{R}, <) = \lambda$, $\text{tp}(\omega_1, \in) = \omega_1$, and $\text{tp}(\omega_1, \in^*) = \omega_1^*$.

If A and B are linearly ordered sets with order types φ and ψ , respectively, then we write $\varphi \preceq \psi$ just in case A is order-isomorphic to a subset of B . If $\varphi \preceq \psi$ we may say that φ is *embeddable* in ψ , or ψ *embeds* φ . If $A \preceq B$ and $B \preceq A$ we say that A and B are *equimorphic* and write $A \sim B$. If there is an order-isomorphism between A and B then we say that A and B are isomorphic and write $A \cong B$. Note that $A \sim B$ does not imply $A \cong B$. In order to see this, set $A = (0, 1)$ and $B = (0, 1) \cup \mathbb{Q}$. Clearly $A \sim B$, but $A \not\cong B$ since B has countable intervals while A does not. This motivates the following definition, which will be used in the sequel.

Definition 1 *Any non-empty set A of real numbers without endpoints is \aleph_1 -dense if between any two elements of A there are exactly \aleph_1 elements of A .*

Let φ be an uncountable linear order type. Then φ is a *real type* if $\varphi \preceq \lambda$, an ω_1 -*type* if $\varphi \preceq \omega_1$ and an ω_1^* -*type* if $\varphi \preceq \omega_1^*$. Real types are essentially characterized by the fact that they are separable while ω_1 -types and ω_1^* -types are characterized by the fact that they are well-ordered and converse well-ordered, respectively. An uncountable linear order that does not embed any uncountable separable linear order, well-order, or converse well-order is an Aronszajn type¹.

Since any uncountable well-order or converse well-order can not be separable we have that $\varphi \not\preceq \omega_1$ and $\varphi \not\preceq \omega_1^*$ for any real type φ . Likewise, since any uncountable suborder of a real type is separable we have that $\varphi \not\preceq \psi$ if ψ is a real type and φ is either an ω_1 -type or ω_1^* -type. Hence, real types, well-orders, converse well-orders, and Aronszajn lines are all pairwise incomparable with respect to \preceq .

¹ In the literature Aronszajn types are also called Specker types.

2.3 The Basis Problem for the Uncountable Linear Orders

The purpose of this section is to give a brief introduction to the basis problem for the uncountable linear orders. In particular, we explain the consistency of a five element basis, which must include a Countryman type and its converse. Further reading and references can be found in Moore [10], and Todorcevic [19].

A basis for the class of uncountable linear orders is any class \mathcal{B} such that for any uncountable linear order L there is a $B \in \mathcal{B}$ such that $\text{tp}(B) \preceq \text{tp}(L)$. Ideally \mathcal{B} would be finite. In this section we will explore as a possible basis for the class of uncountable linear orders the set $\mathcal{B} = \{\omega_1, \omega_1^*, A, C, C^*\}$, where A is any subset of \mathbb{R} with cardinality \aleph_1 , and C is any Countryman line.

For \mathcal{B} to be a basis for the class of uncountable linear orders it must include a basis for each of the real types, ω_1 -types, ω_1^* -types, and Aronszajn types. It is classical result that $\{\omega_1\}$ and $\{\omega_1^*\}$ each form a basis for the uncountable well-orders and converse well-orders, respectively, so it remains to look at the real types and Aronszajn types². We begin by asking if $\{A\}$ can serve as a basis for the real types anytime A is a set of reals of size \aleph_1 . The following theorem of Baumgartner shows that it is consistent that such an A can be a basis for the real types.

Theorem 2 ([2]) *The Proper Forcing Axiom (PFA) implies that all \aleph_1 -dense sets of reals are order-isomorphic³.*

² It will be shown in the next section that Countryman lines are Aronszajn types.

³ This is a modern formulation of Baumgartners theorem.

The fact that extra axiomatics are required for this result, i.e. assumption of the Proper Forcing Axiom, is shown by a classical construction of Sierpiński [15] from which it is routine to prove that under the Continuum Hypothesis there is no basis for the uncountable separable linear orders of cardinality less than $|\mathcal{P}(\mathbb{R})|$.

We can conclude from Theorem 2 that, assuming PFA, \mathcal{B} is a basis for the class of uncountable linear orders if and only if $\{C, C^*\}$ is a basis for the Aronszajn lines, where C is any Countryman order⁴. It has been known for some time that the statement “ $\{C, C^*\}$ is a basis for Aronszajn lines” is equivalent to a combinatorial statement about Aronszajn trees⁵. This combinatorial statement is Theorem 3.

Theorem 3 ([10]) *It is relatively consistent that there is an Aronszajn tree T such that if $K \subseteq T$, then there is an uncountable antichain $X \subseteq T$ such that $\wedge(X)$ ⁶ is either contained in or disjoint from K .*

Assuming PFA Moore was able to prove Theorem 3 in [10] so under PFA it is the case that the uncountable linear orders have a five element basis consisting of ω_1 , ω_1^* , any set of reals of cardinality \aleph_1 , and any Countryman line and its converse.

2.4 Countryman Line Properties

The defining quality of a Countryman line, i.e. its square being the union of countably many chains, gives rise to several properties for which we give the proofs here. As is the case with many interesting mathematical objects, these properties

⁴ Baumgartner observed this in [3].

⁵ See [1], [14] and [20].

⁶ If T is a tree and s and t are incomparable in T , then the meet of s and t , denoted $s \wedge t$, is the restriction $s \upharpoonright \Delta(s, t) = t \upharpoonright \Delta(s, t)$. If X is a subset of T , then $\wedge(X) = \{s \wedge t : s, t \in X\}$.

were known before it was shown that a Countryman line actually exists. For the rest of this section we assume $C = (C, <_C)$ is a Countryman line and let $(A_i)_{i=0}^\infty$ witness the fact that C is Countryman, that is let $(A_i)_{i=0}^\infty$ be such that $C \times C = \bigcup_{i=0}^\infty A_i$, where each A_i is a chain in the product order.

Our first fact is easy to see and says that any uncountable subset of a Countryman type is a Countryman type. By comparison, it is also the case that any uncountable subset of a real type is a real type, any uncountable subset of an ω_1 -type is an ω_1 -type and any uncountable subset of an ω_1^* -type is an ω_1^* -type.

Fact 4 *If $C = (C, <_C)$ is Countryman, then any uncountable suborder of C is Countryman.*

Proof. Let $B \subseteq C$ be uncountable. Then $B \times B = \bigcup_{i=0}^\infty [A_i \cap (B \times B)]$, which is a countable union of chains in the product order. ■

Our next fact states that the converse of Countryman type is a Countryman type. By comparison, it is also the case that the converse of a real type is a real type, however the converse of an ω_1 -type is an ω_1^* -type.

Fact 5 *If $C = (C, <_C)$ is Countryman, then $C^* = (C, >_C)$ is Countryman.*

Proof. Let $(x_1, x_2) \leq_C (x'_1, x'_2)$ be elements of A_i for some i . By definition of \leq_C^* and the product order we have that $(x'_1, x'_2) \leq_C^* (x_1, x_2)$, hence each A_i is a chain under \leq_C^* . Further, since C and C^* have the same underlying set we have $C^* \times C^* = \bigcup_{i=0}^\infty A_i$ so that C^* is Countryman. ■

Our next fact states that a Countryman line cannot embed into its converse. By comparison, it is also the case that an ω_1 -type and ω_1^* -type cannot embed into their converses, however a real type can embed into its converse.

Fact 6 *If $C = (C, <_C)$ is Countryman and $L = (L, <_L)$ embeds into C and into C^* , then L is countable.*

Proof. Let $f : L \rightarrow C$ and $g : L \rightarrow C^*$ be embeddings and set $B = \{(f(x), g(x)) : x \in L\}$. Since f and g are one-to-one functions, B is in one-to-one correspondence with L . Now,

$$f(x) \leq f(x') \Leftrightarrow x \leq x' \Leftrightarrow g(x) \leq^* g(x') \Leftrightarrow g(x) \geq g(x'),$$

so that B is an antichain. Since B is an antichain it can intersect each A_i in at most one point and must therefore be countable. Hence, L is countable. ■

The next three facts establish that Countryman lines are Aronszajn lines. The proofs of Facts 7 and 8 make critical use of the Pigeon Hole Principle for uncountable sets. In particular, any partition of an uncountable set into countably many pieces must have at least one piece with uncountable cardinality⁷. Fact 8 also makes use of the fact that if $\text{tp}(A, <) = \omega_1$ and $a \in A$, then there are uncountably many elements of A greater than a .

Fact 7 *If $C = (C, <_C)$ is Countryman, then C does not embed a real type.*

Proof ([8]). By Fact 4 it suffices to show that C itself is not a real type so suppose towards a contradiction that it is. Then C is separable so there is $D \subseteq C$

⁷ For Fact 7 we only need that there is a piece with more than one element but we will need an uncountable piece for Fact 8.

such that D is countably dense in C . Since for each x in C the set $\{x\} \times C$ is uncountable, the Pigeon Hole Principle tells us for each x in C there exist two elements of C , $y'_x < y''_x$, and an $i_x < \omega$ such that $(x, y'_x), (x, y''_x) \in A_{i_x}$. Also, there is a $d_x \in D$ such that $y'_x < d_x < y''_x$. Since there are only countably many pairs (i_x, d_x) the Pigeon Hole Principle again tells us that there are $x' < x''$ in C such that $i_{x'} = i_{x''} = i$ and $d_{x'} = d_{x''} = d$. Then $(x', y'_{x'})$ and $(x'', y'_{x''})$ are both elements of A_i , even though $(x'', y'_{x''})$ and $(x', y'_{x'})$ are incomparable (since $y''_{x'} > d > y'_{x''}$), a contradiction. ■

Fact 8 *If $C = (C, <_C)$ is Countryman, then C does not embed an ω_1 -type.*

Proof ([8]). By Fact 4 it suffices to show that C itself is not a well-order so suppose towards a contradiction that C is an ω_1 -type. Since $\{x\} \times C$ is uncountable, the Pigeon Hole Principle tells us that for every $x \in C$ there is an $i_x < \omega$ such that $D_x = \{y \in C : (x, y) \in A_{i_x}\}$ is uncountable. Fix $x' \in C$. Since there are uncountably many $x \in C$ such that $x > x'$, the Pigeon Hole Principle again tells us that there must be an $x'' > x'$ such that $i_{x''} = i_{x'} = i$. Find y'' so that $(x'', y'') \in A_i$ and then find $y' \in D_{x'}$ so that $y' > y''$. Then $(x', y'), (x'', y'') \in A_i$ even though (x', y') and (x'', y'') are incomparable, a contradiction. ■

Fact 9 *If $C = (C, <_C)$ is Countryman, then C does not embed an ω_1^* -type.*

Proof. This follows from Facts 5 and 8. ■

3 MINIMAL WALKS

3.1 Introduction

The method of minimal walks was developed by Stevo Todorćević in 1984 and the work circulated as [17]. Soon after others started using the method in their own work and so Todorćević published [18] to present the method in a more explicit form and to demonstrate the failure of the partition relation $\omega_1 \rightarrow [\omega_1]_{\omega_1}^2$. The method involves an analysis of several two place functions on the set of countable ordinals, where said functions are recursively defined using C -sequences. After a brief explanation of von Neumann ordinals in Section 3.3, C -sequences and some of the associated functions will be defined in Sections 3.4 and 3.5.

Of note in Section 3.5 will be the definition of the full lower trace of the walks from β and α . This function is a novel presentation of Todorćević's full lower trace sharing many properties but with a closed form definition that allows it to be used easily in the proof in Section 4.2. This definition of the full lower trace of the walks also has potential interest outside of the construction of a Countryman line, particularly where the recursively defined full lower trace has been used.

3.2 Notation and Definitions

We include here the following definitions related to orderings that will be needed in subsequent sections.

Definition 10 If s and t are sequences of natural numbers we write $s \sqsubseteq t$ and say s is an *initial segment* of t if $\text{dom}(s) \subseteq \text{dom}(t)$ and $s(i) = t(i)$ for all $i \in \text{dom}(s)$.

Definition 11 We use $<_{lex}$ for the *right lexicographic ordering* for finite sequences of natural numbers, that is $s <_{lex} t$ if $t \sqsubset s$ or $s(j) < t(j)$ where j is minimal such that $s(j) \neq t(j)$.

Definition 12 If A and B are sets of ordinals we write $A < B$ if every element of A is less than every element of B .

Definition 13 If A and B are sets of ordinals we write $A \sqsubseteq B$ and say A is an *initial segment* of B if $A \subseteq B$ and $A < B \setminus A$.¹

Definition 14 If $(X, <)$ is a linear order, then the *product order* on $X \times X$ is defined by $(x_1, x_2) \leq (x'_1, x'_2)$ if and only if $x_1 \leq x'_1$ and $x_2 \leq x'_2$.

Finally, we use \frown for the concatenation operator and all counting starts at 0.

3.3 The von Neumann Ordinals

All ordinals in this thesis are von Neumann ordinals, for which we give the definition and some basic facts for readers unfamiliar with the concept. First, we say that a set S is *well-ordered* by the relation $<$ if $(S, <)$ is a linearly ordered set and every nonempty subset of S has a least element. Second, we say a set S is

¹ The notions of initial segment in Definitions 10 and 13 agree if we identify sets of ordinals with strictly increasing sequences of ordinals.

transitive if every element of S is a subset of S^2 . A von Neumann ordinal is defined to be any set α that is transitive and well-ordered by the membership relation \in . Facts 15, 16, and 17 below follow easily from this definition.

Fact 15 *If α is an ordinal then $\alpha \notin \alpha$.*

Fact 16 *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*

Fact 17 *If α and β are ordinal numbers such that $\alpha \subset \beta$ then $\alpha \in \beta$.*

Immediate from these facts is that a von Neumann ordinal α is the set of all ordinals less than α , i.e. a von Neumann ordinal is just the set of its predecessors. This is particularly intuitive for the finite ordinals where we identify the natural number n with the set $\{0, 1, 2, \dots, n - 1\}$. The set of all finite ordinals is denoted ω and the set of all countable ordinals is denoted ω_1 . Those von Neumann ordinals that are of the form $\alpha + 1$ for some ordinal α are called *successor ordinals*. If an ordinal is not a successor ordinal then it is called a *limit ordinal*³.

3.4 Fundamental Sequences

The underlying set of the space of countable ordinals is ω_1 , the least uncountable ordinal. The structure imposed on ω_1 is the usual ordering induced by the membership relation along with a system of sets indexed by the countable ordinals $\langle C_\alpha : \alpha < \omega_1 \rangle$, which has the following two properties:

² Put another way, S is transitive if $s \in t \in S$ implies $s \in S$.

³ Although 0 is technically a limit ordinal it is unique in that it is the only limit ordinal without any predecessors.

(a) $C_{\alpha+1} = \{\alpha\}$.

(b) If α is a nonzero limit ordinal then C_α is a cofinal⁴ subset of α such that for all γ less than α , $|C_\alpha \cap \gamma|$ is finite.

Each C_α will be called a *fundamental sequence* or a *ladder* and a system of fundamental sequences will be called a *C-sequence*, at times denoted by \vec{C} . We note that while *C*-sequences are usually used for climbing up in recursive constructions here they will be used as a tool for walking down from one ordinal to a smaller one. We also note that as a structure (ω_1, \in, \vec{C}) is analogous to the more familiar structure $(\mathbb{N}, <, P)$, the space of finite ordinals with the usual ordering and predecessor function.

3.5 Two Place Functions

Referring to Figure 1 on page 19 will facilitate much of the discussion and many of the definitions in this section. For the remainder of the section we fix a *C*-sequence and nonzero countable ordinals $\alpha < \beta$.

3.5.1 The Steps and Stops of the Walk from β to α

The basic concept when walking from a countable ordinal β to a smaller ordinal α is to find the minimal element of C_β that is greater than or equal to α . This element is the first *step* from β towards α and is denoted β_1 . The greatest element of C_β that is less than α , if there is one, is defined to be the first *stop* in the walk

⁴ A subset C of an ordinal α is *cofinal* in α if for every $\beta \in \alpha$, there is a $\gamma \in C$ such that $\gamma > \beta$.

from β to α and is denoted $\hat{\beta}_1$ (if there is no such element we define $\hat{\beta}_1 = 0$). The next step in the walk is then defined to be the least element of C_{β_1} that is greater than or equal to α and the next stop is the greatest element of C_{β_1} that is less than α . One continues this process until the next step is α itself. This intuitive description is captured formally by the following definitions and facts.

Definition 18 *The i^{th} step from β to α is given by*

$$\begin{aligned}\beta_0 &= \beta, \\ \beta_i &= \min(C_{\beta_{i-1}} \setminus \alpha) \quad (i > 0).\end{aligned}$$

Fact 19 *Since $\beta_i > \beta_{i+1}$ it must be that there is an $l < \omega$ such that $\beta_l = \alpha$, otherwise $(\beta_i)_{i < \omega}$ would be an infinite strictly decreasing set of ordinals.*

Definition 20 *The i^{th} stop from β to α is given by*

$$\begin{aligned}\hat{\beta}_0 &= 0, \\ \hat{\beta}_i &= \max(C_{\beta_{i-1}} \cap \alpha) \quad (i > 0),\end{aligned}$$

where by convention we take $\max \{ \}$ to be 0.

Fact 21 *$\hat{\beta}_i$ and β_i are the unique ordinals in $C_{\beta_{i-1}}$ such that*

$$\hat{\beta}_i < \alpha \leq \beta_i \text{ and } C_{\beta_{i-1}} \cap (\hat{\beta}_i, \beta_i) = \emptyset.$$

3.5.2 The Upper Trace of the Walk from β to α

Definition 22 *For $\alpha < \beta < \omega_1$ the **upper trace** of the walk from α to β is the function*

$$\text{Tr} : [\omega_1]^2 \rightarrow \omega_1^{<\omega}$$

given by

$$\text{Tr}(\beta, \alpha) = \langle \beta_i : i \leq l \rangle,$$

where β_i is as in Definition 18 and l is as in Fact 19.

We will often identify the sequence of ordinals $\text{Tr}(\beta, \alpha)$ with its range.

3.5.3 The Full Code of the Walk from β to α

The *full code* of the walk from β to α is a finite sequence of natural numbers where the i^{th} entry of the sequence is the number of rungs on the ladder C_{β_i} that are below α^5 . Formally the definition is as follows.

Definition 23 *The **full code** of the walk is the function*

$$\rho_0 : [\omega_1]^2 \rightarrow \omega^{<\omega}$$

defined by

$$\begin{aligned} \rho_0(\alpha, \alpha) &= \emptyset, \\ \rho_0(\beta, \alpha)(i) &= |C_{\beta_i} \cap \alpha| \quad (\beta > \alpha), \end{aligned}$$

where β_i is as in Definition 18 and l is as in Fact 19.

The full code of the walk will allow us to define the uncountable linear order $C(\rho_0)$ in Chapter 4. The following fact about ρ_0 will be useful in the sequel.

Fact 24 *Given a system of fundamental sequences and a countable ordinal β the upper trace of the walk from β to α can be recovered from $\rho_0(\beta, \alpha)$.*

⁵ See Figure 1, page 19.

Proof. Let $\langle C_\alpha(i) : i < \omega \rangle$ be the natural enumeration of C_α whenever α is a countable limit ordinal. Then given $\rho_0(\beta, \alpha)$ we can recover β_i using the following recursive construction.

$$\beta_0 = \beta,$$

$$\beta_{i+1} = \begin{cases} \text{the predecessor of } \beta_i & \text{if } \beta_i \text{ is a successor ordinal,} \\ C_{\beta_i}(\rho_0(\alpha, \beta)(i) + 1) & \text{if } \beta_i \text{ is a limit ordinal.} \end{cases}$$

■

3.5.4 The Maximum Stop of the Walk from β to α

The next definition will be used in the proof of several facts that will facilitate the proof in Section 4.2. Referring to Figure 1, page 19, the maximum stop of the walk from β to α , denoted $\lambda(\beta, \alpha)$, is the highest rung of all the rungs that are below α . The formal definition is as follows.

Definition 25 *The **maximum stop** of the walk is the function*

$$\lambda : [\omega_1]^2 \rightarrow \omega_1$$

given by

$$\lambda(\beta, \alpha) = \max_{i \leq l} \hat{\beta}_i,$$

where $\hat{\beta}_i$ is as in Definition 20 and l is as in Fact 19.

3.5.5 The Full Lower Trace of the Walks from β and α

The *full lower trace* of the walks from β and α , denoted $F(\beta, \alpha)$, is the lynchpin of our proof that $C(\rho_0)$ is a Countryman line. Our definition doesn't appear in

Todorcevic’s *Coherent Sequences* but it is a proper subset of the full lower trace as defined there. For completeness and reference we give both definitions here. Throughout the sequel “Todorcevic’s full lower trace” will refer to Definition 26 and “full lower trace”, without qualification, will refer to our Definition 27. In the next section we will prove several facts using our definition that mirror facts that are proven in [21] using Definition 26.

Definition 26 ([21]) *Todorcevic’s full lower trace of the walk from β to α is the function $F : [\omega_1]^2 \rightarrow [\omega_1]^{<\omega}$ defined recursively by*

$$F(\alpha, \alpha) = \{\alpha\},$$

$$F(\alpha, \beta) = F(\alpha, \min(C_\beta \setminus \alpha)) \cup \bigcup_{\xi \in C_\beta \cap \alpha} F(\xi, \alpha).$$

Definition 27 *The full lower trace of the walks from β and α is the set*

$$F(\beta, \alpha) = \{\xi \leq \alpha : \text{Tr}(\beta, \xi) \cap \text{Tr}(\alpha, \xi) = \{\xi\}\}.$$

In order to picture $F(\beta, \alpha)$ intuitively it is illustrative to think of walking up from an ordinal ζ to both β and α ⁶. As one walks up there will be an ordinal ξ from which the two traces $\text{Tr}(\beta, \zeta)$ and $\text{Tr}(\alpha, \zeta)$ will split. That is, while walking up from ζ to ξ the walks from ζ to β and from ζ to α step on exactly the same ordinals and after ξ they never step on the same ordinal again. Our full lower trace is the set of all such ξ . It is this simple closed form definition (modulo the recursive definition of the steps) and the ability to picture the full lower trace that make it attractive for use in further applications.

⁶ See Figure 2, page 20.

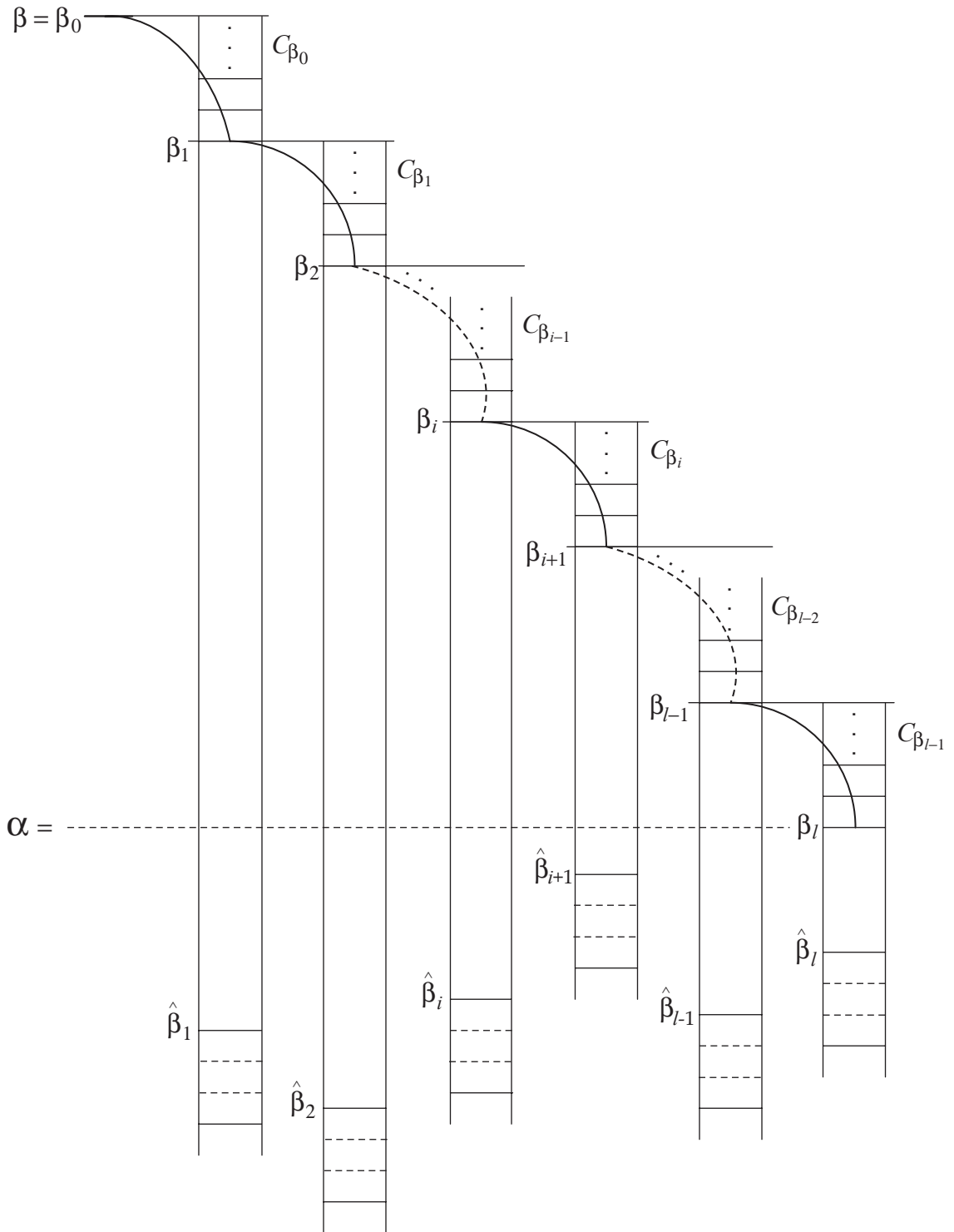


Figure 1. The Walk from β to α

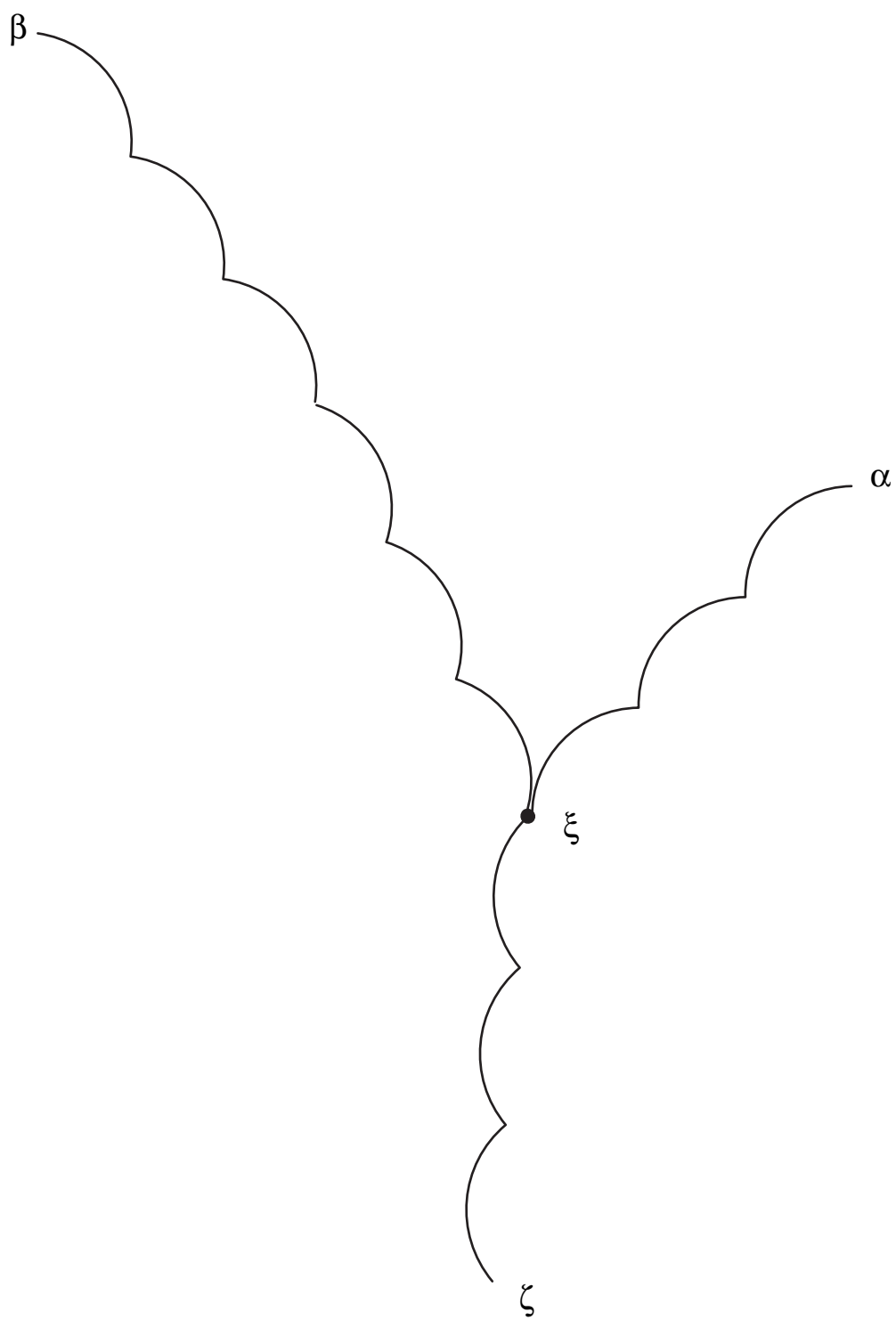


Figure 2. $\xi \in F(\alpha, \beta)$

3.6 Basic Facts

Most of the facts in this section are intuitively clear after a moments reflection on Figure 1, page 19 and Figure 2 on page 20. The proofs of the facts are included both for completeness and to aid in the development of the aforementioned intuition. They are relatively simple using induction on the natural numbers and/or naive set theory.

3.6.1 Facts Needed for the Proof in Section 4.2

Fact 29 below is to the method of minimal walks as $c^2 = a^2 + b^2$ is to the method of geometry. Although it requires two instances of induction to prove, Fact 29 is a really a simple consequence of Definition 18 and Fact 21 and can readily be seen from Figure 1.

Note 28 For Facts 29 and 30 below fix $\zeta < \alpha < \beta < \omega_1$ and set

$$\text{Tr}(\beta, \alpha) = \{\beta = \beta_0 > \beta_1 > \cdots > \beta_l = \alpha\},$$

$$\text{Tr}(\alpha, \zeta) = \{\alpha = \alpha_0 > \alpha_1 > \cdots > \alpha_m = \zeta\},$$

$$\text{Tr}(\beta, \zeta) = \{\beta = \beta'_0 > \beta'_1 > \cdots > \beta'_n = \zeta\},$$

and define the stops for each walk accordingly, that is $\hat{\beta}_i$ is the i^{th} stop in the walk from β to α , $\hat{\alpha}_i$ is the i^{th} stop in the walk from α to ζ , and $\hat{\beta}'_i$ is the i^{th} stop in the walk from β to ζ .

Fact 29 For $\zeta < \alpha < \beta < \omega_1$:

$$\alpha \in \text{Tr}(\beta, \zeta) \text{ if and only if } \text{Tr}(\beta, \zeta) = \text{Tr}(\beta, \alpha) \cup \text{Tr}(\alpha, \zeta).$$

Proof. The reverse implication is immediate since $\alpha \in \text{Tr}(\beta, \alpha)$. For the forward direction refer to Note 28 and assume $\alpha \in \text{Tr}(\beta, \zeta)$. Then there is $k < n$ such that $\beta'_k = \alpha$. We will show by induction on n that

$$i \leq k \Rightarrow \beta'_i = \beta_i \text{ and } k \leq i \leq n \Rightarrow \beta'_i = \alpha_{i-k}.$$

The base case for $i \leq k$ is $\beta'_0 = \beta = \beta_0$. Assume $\beta'_j = \beta_j$, with $j < k$. Since

$$\hat{\beta}'_{j+1} < \zeta < \alpha \leq \beta'_{j+1}$$

and $C_{\beta'_j} = C_{\beta_j}$ we have by Fact 21 that $\beta'_{j+1} = \beta_{j+1}$.

The base case for $k \leq i \leq n$ is $\beta'_k = \alpha = \alpha_{k-k}$. Assume $\beta'_j = \alpha_{j-k}$, with $k \leq j < n$. Then

$$\beta'_{j+1} = \min(C_{\beta'_j} \setminus \zeta) = \min(C_{\alpha_{j-k}} \setminus \zeta) = \alpha_{j-k+1}.$$

The desired conclusion follows. ■

The next fact is telling us that an ordinal α is in the walk from β to ζ if and only if each step of the walk from β to α lies below ζ . Again, this is intuitively clear from Figure 1 and follows from Definition 25 and Facts 21 and 29.

Fact 30 For $\zeta < \alpha < \beta < \omega_1$:

$$\alpha \in \text{Tr}(\beta, \zeta) \text{ if and only if } \lambda(\beta, \alpha) < \zeta.$$

Proof. We use the set up from Note 28. For the forward direction assume $\alpha \in \text{Tr}(\beta, \zeta)$. We have by Fact 29 that for $i \leq l$, $\hat{\beta}_i = \hat{\beta}'_i < \zeta$ and therefore

$$\lambda(\beta, \alpha) = \max_{i \leq m} \hat{\beta}_i < \zeta.$$

For the reverse implication assume $\lambda(\beta, \alpha) < \zeta$. We will show by induction on l that for $i \leq l$, $\beta_i = \beta'_i$. In particular, $\beta'_l = \alpha$ so that $\alpha \in \text{Tr}(\beta, \zeta)$. For the base case we have $\beta'_0 = \beta = \beta_0$. Assume $\beta'_j = \beta_j$, with $j < m$. Since

$$\hat{\beta}_{j+1} \leq \lambda(\beta, \alpha) < \zeta < \alpha \leq \beta_{j+1},$$

and $C_{\beta'_j} = C_{\beta_j}$ we have by Fact 21 that $\beta'_{j+1} = \beta_{j+1}$. The desired conclusion follows. ■

The proof of Fact 31 below follows nicely from Facts 29 and 30 above. Fact 31 says that if you walk from β and from α to a smaller ordinal ζ then you will pass through the least element of the full lower trace of the walks from β and α that is greater than or equal to ζ .

Fact 31 *If $\zeta < \alpha < \beta < \omega_1$ and $\xi = \min(F(\beta, \alpha) \setminus \zeta)$, then*

$$\text{Tr}(\alpha, \zeta) = \text{Tr}(\alpha, \xi) \cup \text{Tr}(\xi, \zeta) \text{ and } \text{Tr}(\beta, \zeta) = \text{Tr}(\beta, \xi) \cup \text{Tr}(\xi, \zeta).$$

Proof. Set $\delta = \max(\text{Tr}(\alpha, \zeta) \cap \text{Tr}(\beta, \zeta))$. Then by Fact 29 we have

$$\text{Tr}(\alpha, \zeta) = \text{Tr}(\alpha, \delta) \cup \text{Tr}(\delta, \zeta) \text{ and } \text{Tr}(\beta, \zeta) = \text{Tr}(\beta, \delta) \cup \text{Tr}(\delta, \zeta).$$

By the maximality of δ , $\text{Tr}(\alpha, \delta) \cap \text{Tr}(\beta, \delta) = \{\delta\}$ and since $\delta \geq \zeta$ we have $\delta \in F(\beta, \alpha) \setminus \zeta$. Now suppose towards a contradiction that there is $\delta' \in F(\beta, \alpha) \setminus \zeta$ such that $\delta' < \delta$. Applying Fact 30 we have

$$\lambda(\alpha, \delta) < \zeta \leq \delta' \text{ and } \lambda(\beta, \delta) < \zeta \leq \delta'.$$

We can then apply Fact 30 again to obtain $\delta \in \text{Tr}(\alpha, \delta') \cap \text{Tr}(\beta, \delta')$, but then $\{\delta, \delta'\} \subseteq \text{Tr}(\alpha, \delta') \cap \text{Tr}(\beta, \delta')$, contrary to our choice of δ' . Thus, there is no $\delta' \in F(\beta, \alpha) \setminus \zeta$ less than δ so $\delta = \xi$ and we are done. ■

Fact 32 follows easily from the previous two facts and mirrors Lemma 1.9 in [21], where the same fact is established for Todorčević's full lower trace.

Fact 32 *If $\zeta \leq \alpha < \beta < \omega_1$ and $\xi = \min(F(\beta, \alpha) \setminus \zeta)$, then*

$$\rho_0(\alpha, \zeta) = \rho_0(\alpha, \xi) \frown \rho_0(\xi, \zeta) \text{ and } \rho_0(\beta, \zeta) = \rho_0(\beta, \xi) \frown \rho_0(\xi, \zeta).$$

Proof. By Fact 31 we have $\text{Tr}(\alpha, \zeta) = \text{Tr}(\alpha, \xi) \cup \text{Tr}(\xi, \zeta)$. Set

$$\text{Tr}(\alpha, \xi) = \{\alpha = \alpha_0 > \alpha_1 > \cdots > \alpha_l = \xi\},$$

$$\text{Tr}(\xi, \zeta) = \{\xi = \xi_0 > \xi_1 > \cdots > \xi_m = \zeta\},$$

$$\text{Tr}(\alpha, \zeta) = \{\alpha = \alpha'_0 > \alpha'_1 > \cdots > \alpha'_n = \zeta\}.$$

By Fact 30 $\lambda(\alpha, \xi) \leq \zeta$, hence by Definition 23 for $i < l$,

$$\rho_0(\alpha, \zeta)(i) = |C_{\alpha'_{i+1}} \cap \zeta| = |C_{\alpha_{i+1}} \cap \xi| = \rho_0(\alpha, \xi)(i).$$

On the other hand, for $l \leq i < n$

$$\rho_0(\alpha, \zeta)(i) = |C_{\alpha'_{i+1}} \cap \zeta| = |C_{\xi_{i-l}} \cap \zeta| = \rho_0(\xi, \zeta)(i-l).$$

Thus,

$$\begin{aligned} \rho_0(\alpha, \zeta) &= \\ \rho_0(\alpha, \zeta)(0) \frown \cdots \frown \rho_0(\alpha, \zeta)(l-1) \frown \rho_0(\alpha, \zeta)(l) \frown \cdots \frown \rho_0(\alpha, \zeta)(n-1) &= \\ \rho_0(\alpha, \xi)(0) \frown \cdots \frown \rho_0(\alpha, \xi)(l-1) \frown \rho_0(\xi, \zeta)(0) \frown \cdots \frown \rho_0(\xi, \zeta)(m-1) &= \\ &= \rho_0(\alpha, \xi) \frown \rho_0(\xi, \zeta) \end{aligned}$$

This gives the left hand conjunct of the conclusion. Replacing α everywhere with β gives the right hand conjunct, so we are done. ■

The fact that $F(\beta, \alpha)$ is finite is not immediately clear from either definition or picture. Indeed, if β and α are larger than ω then there are infinitely many walks from β to a smaller ordinal and infinitely many walks from α to a smaller ordinal. Hence, it is conceivable that there are infinitely many walks from β and α that meet in distinct ordinals, in which case $F(\beta, \alpha)$ would be an infinite set. The proof that this is not the case demonstrates a general method of proof that is often used in the method of minimal walks.

Fact 33 *If $\alpha < \beta < \omega_1$, then $F(\beta, \alpha)$ is finite.*

Proof. Suppose $F(\beta, \alpha)$ is infinite and let δ be a limit point of $F(\beta, \alpha)$. Set $\delta' = \max(\lambda(\alpha, \delta), \lambda(\beta, \delta))$ and suppose $\xi \in (\delta', \delta)$. Then by Fact 30, $\delta \in \text{Tr}(\alpha, \xi)$ and $\delta \in \text{Tr}(\beta, \xi)$, so $\{\xi, \delta\} \subseteq \text{Tr}(\alpha, \xi) \cap \text{Tr}(\beta, \xi)$. Hence, $\xi \notin F(\beta, \alpha)$ for any $\xi \in (\delta', \delta)$, contradicting the choice of δ . ■

Fact 34 will allow us to identify a countable ordinal β with the one-place function $\rho_0(\beta, \cdot) : \beta \rightarrow \omega^{<\omega}$, i.e. a β sequence of finite sequences of integers.

Fact 34 *If $\beta < \omega_1$, then $\rho_0(\beta, \cdot) : \beta \rightarrow \omega^{<\omega}$ is a one-to-one function.*

Proof. Suppose $\alpha, \alpha' < \beta$ and $\rho_0(\beta, \alpha) = \rho_0(\beta, \alpha')$. Let $\text{Tr}(\beta, \alpha) = \{\beta_0, \dots, \beta_l\}$ and $\text{Tr}(\beta, \alpha') = \{\beta'_0, \dots, \beta'_l\}$. We will prove by induction on $l = |\rho_0(\beta, \alpha)| = |\rho_0(\beta, \alpha')|$ that $\beta_i = \beta'_i$. In particular, $\alpha = \beta_l = \beta'_l = \alpha'$. For the base case we have $\beta_0 = \beta = \beta'_0$. Assume $\beta_j = \beta'_j$ for $j < l$. Then $C_{\beta_j} = C_{\beta'_j}$ so from Fact 24

$$\beta_{j+1} = C_{\beta_j}(\rho_0(\beta, \alpha)(j) + 1) = C_{\beta'_j}(\rho_0(\beta, \alpha')(j) + 1) = \beta'_{j+1}.$$

The desired conclusion follows. ■

3.6.2 Further Facts

We include the remaining facts for the interested reader even though they aren't needed for the proof in Section 4.2. These facts demonstrate some of the useful properties of $\rho_0(\beta, \alpha)$ and $F(\beta, \alpha)$ and the proofs help to develop an intuition as to how to use the method of minimal walks in the space of countable ordinals.

Fact 35 For $\zeta < \alpha < \beta < \omega_1$:

$$\alpha \in \text{Tr}(\beta, \zeta) \text{ if and only if } \rho_0(\beta, \alpha) \sqsubseteq \rho_0(\beta, \zeta).$$

Proof. We use the set up from Note 28. For the forward direction assume $\alpha \in \text{Tr}(\beta, \zeta)$. Since $\zeta < \alpha$ we have $|\rho_0(\beta, \alpha)| < |\rho_0(\beta, \zeta)|$. Further, from Facts 21 and 29 we have for $i < l$,

$$\begin{aligned} \rho_0(\beta, \alpha)(i) &= |C_{\beta_i} \cap \alpha| = \left| \left\{ \xi \in C_{\beta_i} : \xi \leq \hat{\beta}_{i+1} \right\} \right| \\ &= \left| \left\{ \xi \in C_{\beta'_i} : \xi \leq \hat{\beta}'_{i+1} \right\} \right| = |C_{\beta'_i} \cap \zeta| = \rho_0(\beta, \zeta)(i), \end{aligned}$$

so $\rho_0(\beta, \alpha) \sqsubseteq \rho_0(\beta, \zeta)$.

For the reverse implication assume $\rho_0(\beta, \alpha) \sqsubseteq \rho_0(\beta, \zeta)$. We will show by induction on l that for $i \leq l$, $\beta'_i = \beta_i$. In particular, $\beta'_l = \alpha$ so that $\alpha \in \text{Tr}(\beta, \zeta)$. For the base case we have $\beta'_0 = \beta = \beta_0$. Assume $\beta'_j = \beta_j$, for $j < l$ and let $\langle C_\beta(i) : i < \omega \rangle$ be the natural enumeration of C_β for $\beta \in \omega_1$. Then by Fact 24

$$\beta_{j+1} = C_{\beta_j}(\rho_0(\beta, \alpha)(j) + 1) = C_{\beta'_j}(\rho_0(\beta, \zeta)(j) + 1) = \beta'_{j+1}.$$

The desired conclusion follows. ■

The following is a generalization of Fact 32 above.

Fact 36 *If $\zeta < \alpha < \beta < \omega_1$ and $\lambda(\beta, \alpha) < \zeta$, then*

$$\rho_0(\beta, \zeta) = \rho_0(\beta, \alpha) \frown \rho_0(\alpha, \zeta).$$

Proof. Using the set up from Note 28 assume $\lambda(\beta, \alpha) < \zeta$. By Facts 30 and 35 $\rho_0(\beta, \alpha) \sqsubseteq \rho_0(\beta, \zeta)$, so that $\rho_0(\beta, \zeta)(i) = \rho_0(\beta, \alpha)(i)$ for $i < l$. On the other hand, by Fact 29

$$\rho_0(\beta, \zeta)(i) = |C_{\beta_i} \cap \zeta| = |C_{\alpha_{i-m}} \cap \zeta| = \rho_0(\alpha, \zeta)(i-l).$$

for $l \leq i < n$. The desired conclusion follows. ■

For the proof of the next fact note that if j is minimal such that $\rho_0(\beta, \alpha)(j) \neq \rho_0(\beta, \zeta)(j)$ then the $j+1^{\text{st}}$ step from β to ζ must be between ζ and α , else the steps would agree (since $C_{\beta'_j} = C_{\beta_j}$).

Fact 37 *If $\zeta < \alpha < \beta < \omega_1$, then $\rho_0(\beta, \zeta) <_{\text{lex}} \rho_0(\beta, \alpha)$*

Proof. Set $\text{Tr}(\beta, \alpha) = \{\beta_0, \dots, \beta_l\}$ and $\text{Tr}(\beta, \zeta) = \{\beta'_0, \dots, \beta'_n\}$. If $\rho_0(\beta, \alpha) \sqsubseteq \rho_0(\beta, \zeta)$ then we are done. Otherwise, let j be minimal such that $\rho_0(\beta, \alpha)(j) \neq \rho_0(\beta, \zeta)(j)$. Then since $C_{\beta'_j} = C_{\beta_j}$ it must be that

$$\zeta \leq \beta'_{j+1} < \alpha \leq \beta_{j+1},$$

so

$$\rho_0(\beta, \alpha)(j) = |C_{\beta_j} \cap \alpha| \geq |(C_{\beta_j} \cap \zeta) \cup \{\beta_{j+1}\}| > |C_{\beta_j} \cap \zeta| = \rho_0(\beta, \zeta)(j).$$

Hence, $\rho_0(\beta, \zeta) <_{\text{lex}} \rho_0(\beta, \alpha)$. ■

The following fact is a rewording of what was said in the paragraph following Definition 27 on page 18 and is immediately clear from Figure 2 on, page 20.

Fact 38 *If $\zeta < \alpha < \beta < \omega_1$ and $\xi = \max(\text{Tr}(\beta, \zeta) \cap \text{Tr}(\alpha, \zeta))$, then*

$$\text{Tr}(\beta, \zeta) \cap \text{Tr}(\alpha, \zeta) = \text{Tr}(\xi, \zeta).$$

Proof. The first equality below follows from Fact 29, the second equality by factoring, the third by the maximality of ξ , and the fourth by Definition 22.

$$\begin{aligned} \text{Tr}(\beta, \zeta) \cap \text{Tr}(\alpha, \zeta) &= (\text{Tr}(\beta, \xi) \cup \text{Tr}(\xi, \zeta)) \cap (\text{Tr}(\alpha, \xi) \cup \text{Tr}(\xi, \zeta)) \\ &= (\text{Tr}(\beta, \xi) \cap \text{Tr}(\alpha, \xi)) \cup \text{Tr}(\xi, \zeta) = \{\xi\} \cup \text{Tr}(\xi, \zeta) = \text{Tr}(\xi, \zeta). \end{aligned}$$

This completes the proof. ■

The next fact follows immediately from the previous one by applying Definition 13. It will be used to prove Fact 40 below.

Fact 39 *If $\zeta < \alpha < \beta < \omega_1$ and $\xi = \max(\text{Tr}(\beta, \zeta) \cap \text{Tr}(\alpha, \zeta))$, then*

$$\text{Tr}(\xi, \zeta) \sqsubseteq \text{Tr}(\beta, \zeta) \text{ and } \text{Tr}(\xi, \zeta) \sqsubseteq \text{Tr}(\alpha, \zeta).$$

Proof. It follows from Fact 38 that

$$\text{Tr}(\beta, \zeta) = \text{Tr}(\beta, \xi) \cup \text{Tr}(\xi, \zeta) \text{ and } \text{Tr}(\alpha, \zeta) = \text{Tr}(\alpha, \xi) \cup \text{Tr}(\xi, \zeta)$$

so that

$$\text{Tr}(\beta, \zeta) \setminus \text{Tr}(\xi, \zeta) = \text{Tr}(\beta, \xi) \text{ and } \text{Tr}(\alpha, \zeta) \setminus \text{Tr}(\xi, \zeta) = \text{Tr}(\alpha, \xi).$$

The desired conclusion now follows from the fact that for each $\delta \in \text{Tr}(\xi, \zeta)$ and each $\delta' \in \text{Tr}(\beta, \xi) \cup \text{Tr}(\alpha, \xi)$ we have $\delta \leq \xi \leq \delta'$. ■

The next fact is what gives our full lower trace of the walks potential to be used in other applications found in Todorcevic's *Coherent Sequences*. It is illuminating to compare the following proof with the proof of Lemma 1.8 in [21]. Figure 3 on page 30 illustrates our proof.

Fact 40 *If $\zeta < \alpha < \beta < \omega_1$, then $F(\alpha, \zeta) \subseteq F(\beta, \zeta) \cup F(\beta, \alpha)$.*

Proof. We will prove the contrapositive, that is, if $\xi \notin F(\beta, \zeta)$ and $\xi \notin F(\beta, \alpha)$, then $\xi \notin F(\alpha, \zeta)$. To see this suppose $\xi < \zeta$ and $\xi \notin F(\beta, \zeta)$ and $\xi \notin F(\beta, \alpha)$. Let

$$\xi' = \max(\text{Tr}(\beta, \xi) \cap \text{Tr}(\zeta, \xi)) \text{ and } \xi'' = \max(\text{Tr}(\beta, \xi) \cap \text{Tr}(\alpha, \xi)).$$

By Fact 39,

$$\text{Tr}(\xi', \xi) \sqsubseteq \text{Tr}(\beta, \xi) \text{ and } \text{Tr}(\xi'', \xi) \sqsubseteq \text{Tr}(\beta, \xi).$$

Hence, if $\xi' > \xi''$, then again by Fact 39

$$\text{Tr}(\xi'', \xi) \sqsubseteq \text{Tr}(\xi', \xi) \sqsubseteq \text{Tr}(\zeta, \xi) \Rightarrow \xi'' \in \text{Tr}(\zeta, \xi)$$

and therefore

$$\{\xi'', \xi\} \subseteq (\text{Tr}(\zeta, \xi) \cap \text{Tr}(\alpha, \xi)) \Rightarrow \xi \notin F(\alpha, \zeta).$$

On the other hand, if $\xi' < \xi''$, then by Fact 39

$$\text{Tr}(\xi', \xi) \sqsubseteq \text{Tr}(\xi'', \xi) \sqsubseteq \text{Tr}(\alpha, \xi) \Rightarrow \xi' \in \text{Tr}(\alpha, \xi)$$

and therefore

$$\{\xi, \xi'\} \subseteq \text{Tr}(\zeta, \xi) \cap \text{Tr}(\alpha, \xi) \Rightarrow \xi \notin F(\alpha, \zeta).$$

■

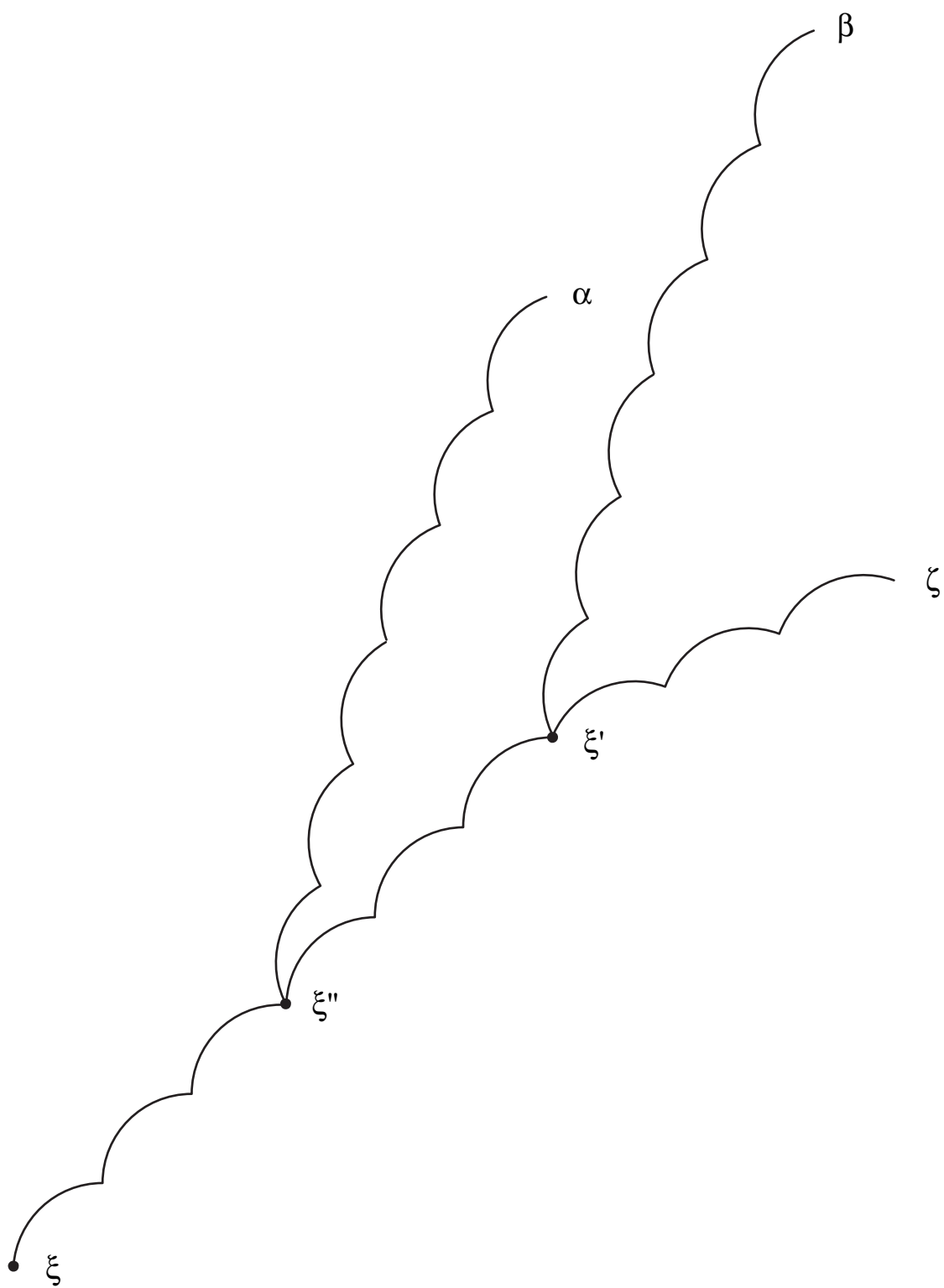


Figure 3. Fact 40: $\xi' \geq \xi''$

4 A PROOF THAT $C(\rho_0)$ IS COUNTRYMAN

4.1 Introduction

Using a variant of a well-known construction of an Aronszajn tree Shelah was the first to construct a Countryman line¹. Following this, Todorćević developed his method of minimal walks while discovering a new construction². In doing so he was able to generalize to Theorem 41 below. Before stating the theorem we give some background definitions. Let $a : [\omega_1]^2 \rightarrow \omega$ be a mapping and let a_α be short for $a(\cdot, \alpha)$. We write $a_\alpha =^* a_\beta \upharpoonright \alpha$ just in case the set $\{\xi < \alpha : a_\alpha(\xi) \neq a_\beta(\xi)\}$ is finite. Finally, $(T(a), <_l)$ represents the tree $\{a_\beta \upharpoonright \alpha : \alpha < \beta < \omega_1\}$ ordered lexicographically.

Theorem 41 ([18]) *Suppose a_α is finite-to-one and $a_\alpha =^* a_\beta \upharpoonright \alpha$ for all $\alpha < \beta < \omega_1$. Then $(T(a), <_l)$ is a Countryman type.*

From this theorem Todorćević is able to show that several trees defined using functions from the method of minimal walks give rise to Countryman orders under their lexicographic ordering. Indeed, the functions that he uses are all *coherent* in the sense that they have the properties of the mapping a above. These mappings and more can be found in his *Coherent Sequences* [21] chapter of the forthcoming Handbook of Set Theory.

¹ See [13].

² See [18].

However, the function ρ_0 is not coherent. In fact, it is an open problem in ZFC as to whether $T(\rho_0)$ is coherent or if $C(\rho_0)^3$ is a lexicographic order on a coherent sequence. Hence, in order to show that the lexicographic ordering of $T(\rho_0)$ is Countryman Todorcevic essentially modifies the proof of Theorem 41 replacing a critical finite set with his full lower trace. We won't reproduce the details of that proof here⁴ but mention that in the section below our definition of the full lower trace also plays the role of this critical finite set.

4.2 The Proof

In order to define $C(\rho_0)$ we first need to define the order $<_{\rho_0}$ on the set ω_1 .

Definition 42 *The ordering $<_{\rho_0}$ is induced by the right lexicographic order $<_{lex}$ and is defined as follows for $\alpha, \beta \in \omega_1$:*

$$\alpha <_{\rho_0} \beta \text{ if and only if } \rho_0(\alpha, \delta) <_{lex} \rho_0(\beta, \delta),$$

where $\delta = \Delta(\alpha, \beta) = \min\{\zeta < \min\{\alpha, \beta\} : \rho_0(\alpha, \zeta) \neq \rho_0(\beta, \zeta)\}$.

Definition 43 $C(\rho_0) = (\omega_1, <_{\rho_0})$.

Definition 44 allows us to partition $\omega_1 \times \omega_1$ into countably many sets.

Definition 44 *Define $\sigma : [\omega_1]^2 \rightarrow ((\omega^{<\omega})^2)^{<\omega}$ by*

$$\sigma(\alpha, \beta)(i) = (\rho_0(\alpha, \xi_i), \rho_0(\beta, \xi_i)),$$

where $(\xi_i)_{i=0}^n$ is the increasing enumeration of $F(\beta, \alpha)$.

³ See Definition 43.

⁴ See [18] or [4].

The following define subsets of $\omega_1 \times \omega_1$ using the function σ above.

Definition 45 For $\tau \in ((\omega^{<\omega})^2)^{<\omega}$ define $\Gamma_\tau = \{(\alpha, \beta) : \alpha < \beta < \omega_1 \text{ and } \sigma(\alpha, \beta) = \tau\}$.

Definition 46 For $\tau \in ((\omega^{<\omega})^2)^{<\omega}$ define $\Gamma_\tau^* = \{(\alpha, \beta) : \beta < \alpha < \omega_1 \text{ and } \sigma(\alpha, \beta) = \tau\}$.

The fact that $C(\rho_0)$ is Countryman follows easily from the following lemma.

Lemma 47 Suppose

$$(\alpha, \beta), (\alpha', \beta') \in \Gamma_\tau,$$

then

$$\alpha <_{\rho_0} \alpha' \text{ if and only if } \beta <_{\rho_0} \beta'.$$

Proof. Since $(\alpha, \beta), (\alpha', \beta') \in \Gamma_\tau$ it must be that $|F(\alpha, \beta)| = |F(\alpha', \beta')|$ so let $(\xi_i)_{i=0}^n$ and $(\xi'_i)_{i=0}^n$ be the respective increasing enumerations of $F(\alpha, \beta)$ and $F(\alpha', \beta')$. Further, let j be minimal such that $\xi_j \neq \xi'_j$ and assume without loss of generality that $\xi_j < \xi'_j$. Then, by assumption, for all $i \leq n$ we have

$$\rho_0(\alpha, \xi_i) = \rho_0(\alpha', \xi'_i) \text{ and } \rho_0(\beta, \xi_i) = \rho_0(\beta', \xi'_i). \quad (a)$$

For ease of reference we reproduce here Fact 32:

If $\zeta \leq \alpha < \beta < \omega_1$ and $\xi = \min(F(\beta, \alpha) \setminus \zeta)$, then

$$\rho_0(\alpha, \zeta) = \rho_0(\alpha, \xi) \frown \rho_0(\xi, \zeta) \text{ and } \rho_0(\beta, \zeta) = \rho_0(\beta, \xi) \frown \rho_0(\xi, \zeta). \quad (b)$$

In what follows we will make free and frequent use of (a) and (b).

Our first task is to show that $\Delta(\alpha, \alpha') = \Delta(\beta, \beta')$ and that their common value lies in the interval $(\xi_{j-1}, \xi_j]$. Towards that end we have from Definition 11

that $\rho_0(\xi'_j, \xi_j) \neq \emptyset$ so by (a) and (b),

$$\begin{aligned}\rho_0(\alpha', \xi_j) &= \rho_0(\alpha', \xi'_j) \frown \rho_0(\xi'_j, \xi_j) \\ &= \rho_0(\alpha, \xi_j) \frown \rho_0(\xi'_j, \xi_j) \neq \rho_0(\alpha, \xi_j)\end{aligned}$$

and

$$\begin{aligned}\rho_0(\beta', \xi_j) &= \rho_0(\beta', \xi'_j) \frown \rho_0(\xi'_j, \xi_j) \\ &= \rho_0(\beta, \xi_j) \frown \rho_0(\xi'_j, \xi_j) \neq \rho_0(\beta, \xi_j).\end{aligned}$$

Hence,

$$\Delta(\alpha, \alpha'), \Delta(\beta, \beta') \leq \xi_j. \quad (c)$$

Next, suppose $\zeta \leq \xi'_{j-1} = \xi_{j-1}$. Then there is $k < j$ such that

$$\xi'_{k-1} = \xi_{k-1} < \zeta \leq \xi_k = \xi'_k < \xi_j$$

and therefore

$$\min(F(\beta, \alpha) \setminus \zeta) = \xi_k = \xi'_k = \min(F(\beta', \alpha') \setminus \zeta).$$

So by (a) and (b)

$$\begin{aligned}\rho_0(\alpha, \zeta) &= \rho_0(\alpha, \xi_k) \frown \rho_0(\xi_k, \zeta) \\ &= \rho_0(\alpha', \xi'_k) \frown \rho_0(\xi'_k, \zeta) = \rho_0(\alpha', \zeta),\end{aligned}$$

and

$$\begin{aligned}\rho_0(\beta, \zeta) &= \rho_0(\beta, \xi_k) \frown \rho_0(\xi_k, \zeta) \\ &= \rho_0(\beta', \xi'_k) \frown \rho_0(\xi'_k, \zeta) = \rho_0(\beta', \zeta).\end{aligned}$$

Thus,

$$\Delta(\alpha, \alpha'), \Delta(\beta, \beta') > \xi_{j-1}. \quad (d)$$

Thus, by (c) and (d)

$$\Delta(\alpha, \alpha'), \Delta(\beta, \beta') \in (\xi_{j-1}, \xi_j]. \quad (e)$$

Now, set $\Delta(\alpha, \alpha') = \delta_1$ and $\Delta(\beta, \beta') = \delta_2$ and notice that by (e) we have

$$\xi_j = \min(F(\beta, \alpha) \setminus \delta_1) \text{ and } \xi'_j = \min(F(\beta', \alpha') \setminus \delta_2). \quad (f)$$

Further, $\rho_0(\alpha, \delta_1) \neq \rho_0(\alpha', \delta_1)$, so by (b)

$$\rho_0(\alpha, \xi_j) \frown \rho_0(\xi_j, \delta_1) \neq \rho_0(\alpha', \xi'_j) \frown \rho_0(\xi'_j, \delta_1),$$

and therefore by (a), $\rho_0(\xi_j, \delta_1) \neq \rho_0(\xi'_j, \delta_1)$. This in turn implies

$$\rho_0(\beta, \xi_j) \frown \rho_0(\xi_j, \delta_1) \neq \rho_0(\beta', \xi'_j) \frown \rho_0(\xi'_j, \delta_1),$$

so that $\rho_0(\beta, \delta_1) \neq \rho_0(\beta', \delta_1)$ and thus $\delta_2 \leq \delta_1$. A symmetrical argument shows that $\delta_1 \leq \delta_2$ and therefore $\delta_1 = \delta_2$.

We are now ready to finish the proof. Let $\delta = \delta_1 = \delta_2$. By (f) and (b) we have

$$\rho_0(\alpha, \delta) = \rho_0(\alpha, \xi_j) \frown \rho_0(\xi_j, \delta),$$

$$\rho_0(\alpha', \delta) = \rho_0(\alpha', \xi'_j) \frown \rho_0(\xi'_j, \delta),$$

and

$$\rho_0(\beta, \delta) = \rho_0(\beta, \xi_j) \frown \rho_0(\xi_j, \delta),$$

$$\rho_0(\beta', \delta) = \rho_0(\beta', \xi'_j) \frown \rho_0(\xi'_j, \delta).$$

Therefore, by (a)

$\rho_0(\alpha, \delta) <_{lex} \rho_0(\alpha', \delta)$ if and only if

$\rho_0(\xi_j, \delta) <_{lex} \rho_0(\xi'_j, \delta)$ if and only if

$\rho_0(\beta, \delta) <_{lex} \rho_0(\beta', \delta)$.

The desired conclusion follows by the definition of $<_{\rho_0}$. ■

Theorem 48 $C(\rho_0)$ is a Countryman line.

Proof. Let $D = \{(\alpha, \alpha) : \alpha \in \omega_1\}$ and set $X = ((\omega^{<\omega})^2)^{<\omega}$. Clearly,

$$\omega_1 \times \omega_1 = D \cup \left(\bigcup_{\tau \in X} \Gamma_\tau \right) \cup \left(\bigcup_{\tau \in X} \Gamma_\tau^* \right),$$

and since $|X| = \omega$ we have decomposed the Cartesian square of ω_1 into a countable union of sets. Since D is a chain it remains to show that Γ_τ and Γ_τ^* are chains. Of course, it is the content of Lemma 47 that Γ_τ is a chain and the fact that Γ_τ^* is a chain follows symmetrically by replacing Γ_τ with Γ_τ^* in the proof of the lemma. ■

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